



Section 9.3A

Tests About a Population Mean

Objectives

- ✓ CHECK conditions for carrying out a test about a population mean.
- ✓ CONDUCT a one-sample t test about a population mean.
- ✓ CONSTRUCT a confidence interval to draw a conclusion for a two-sided test about a population mean.



- **Introduction**

Confidence intervals and significance tests for a population proportion p are based on z -values from the standard Normal distribution.

Inference about a population mean μ uses a t distribution with $n - 1$ degrees of freedom, **except** in the rare case when the population standard deviation σ is known.

We learned how to construct confidence intervals for a population mean.

Now we'll examine the details of testing a claim about an unknown parameter μ .

• The One-Sample t Test

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When the conditions are met, we can test a claim about a population mean μ using a **one-sample t test**.

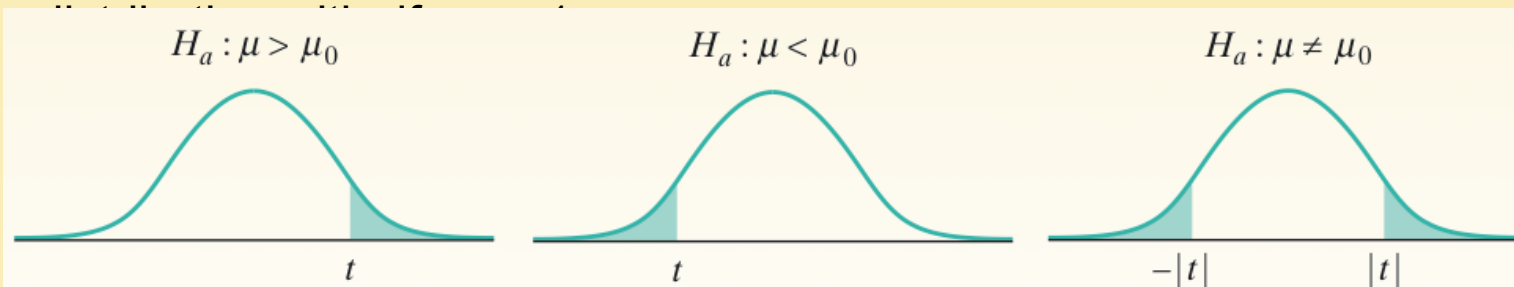
Use this test only when (1) the population distribution is Normal or the sample is large ($n \geq 30$), and (2) the population is at least 10 times as large as the sample.

One-Sample t Test

Choose an SRS of size n from a large population that contains an unknown mean μ . To test the hypothesis $H_0 : \mu = \mu_0$, compute the one-sample t statistic

$$t = \frac{\bar{x} - \mu_0}{s_x / \sqrt{n}}$$

Find the P -value by calculating the probability of getting a t statistic this large or larger in the direction specified by the alternative hypothesis H_a in a t -



Using Table B to find P-Values for a T-distribution

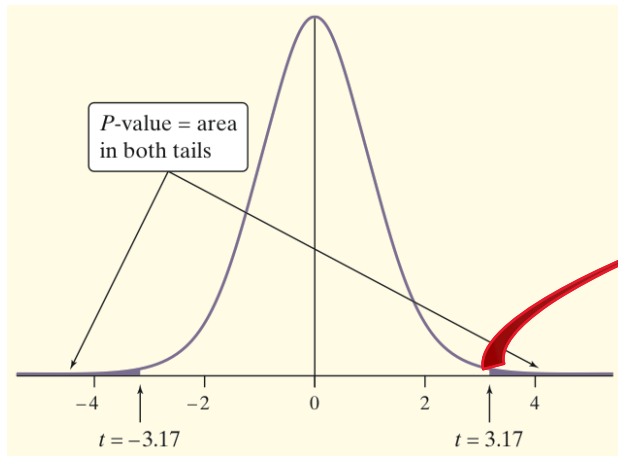


- To find P-values for a significance for t distributions use Table B if you do not have a calculator.
- Table B includes probabilities for degrees of freedom from 1 to 30 and then skips to $df = 40, 50, 60, 80, 100,$ and 1000.
- The bottom row gives probabilities for $df = \infty$, which corresponds to the standard Normal curve.
- Note: If the df you need isn't provided in Table B, use the next lower df that is available.
- Table B shows probabilities only for positive values of t . To find a P-value for a negative value of t , we use the symmetry of the t distributions.

EXAMPLE:

Suppose you were performing a test of $H_0: \mu = 5$ versus $H_a: \mu \neq 5$ based on a sample size of $n = 37$ and obtained $t = -3.17$.

- Since this is a 2-sided test, you are interested in the probability of getting a value of t less than -3.17 or greater than 3.17 .
- Due to the symmetric shape of the density curve, $P(t \leq -3.17) = P(t \geq 3.17)$.
- Since Table B shows only positive t -values, we must focus on $t = 3.17$.



df	Upper-tail probability p		
	.005	.0025	.001
29	2.756	3.038	3.396
30	2.750	3.030	3.385
40	2.704	2.971	3.307
	99%	99.5%	99.8%
	Confidence level C		

- Since $df = 37 - 1 = 36$ is not available on the table, move across the $df = 30$ row, notice that $t = 3.17$ falls between 3.030 & 3.385.
- The corresponding "Upper-tail probability p " is between 0.0025 and 0.001.
- For this 2-sided test, the corresponding P-value would be between $2(0.001) = 0.002$ and $2(0.0025) = 0.005$.

■ Calculations: Test statistic and P-value



When performing a significance test, we do calculations assuming that the null hypothesis H_0 is true. The test statistic measures how far the sample result diverges from the parameter value specified by H_0 , in standardized units. As before,

$$\text{test statistic} = \frac{\text{statistic} - \text{parameter}}{\text{standard deviation of statistic}}$$

For a test of $H_0: \mu = \mu_0$, our statistic is the sample mean. Its standard deviation is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

Because the population standard deviation σ is usually unknown (4th condition), we use the sample standard deviation s_x in its place. The resulting test statistic has the standard error of the sample mean in the denominator

$$t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}}$$

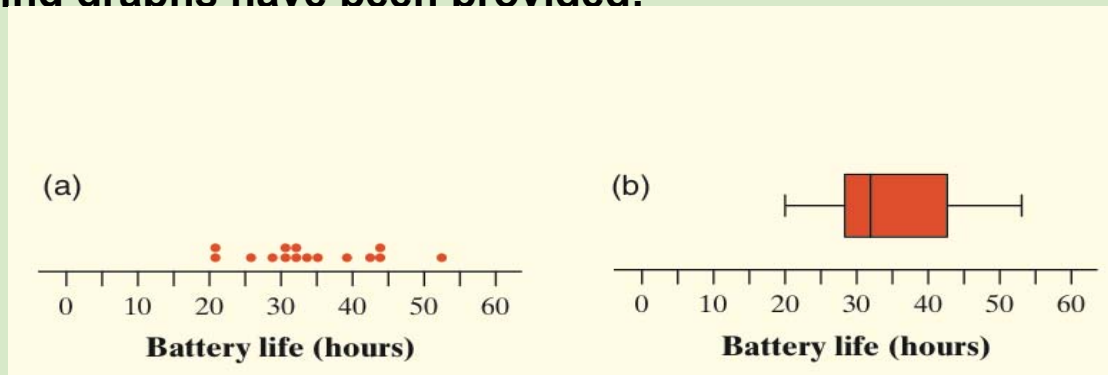
When the Normal condition is met, this statistic has a t distribution with $n - 1$ degrees of freedom.

■ **The One-Sample t Test for μ – Example #1 “AAA Batteries”**

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EXAMPLE #1: A company claimed to have developed a new AAA battery that lasts longer than its regular AAA batteries. Based on years of experience, the company knows that its regular AAA batteries last for 30 hours of continuous use, on average. An SRS of 15 new batteries lasted an average of 33.9 hours with a standard deviation of 9.8 hours. Do these data give *convincing evidence* that the new batteries last longer on average?

The following graphs have been provided:



1) State Hypotheses and Sketch Graph:

2) Check Conditions:

3) Calculations: Test statistic and P-value:

4) Conclusion:

■ *The One-Sample t Test for μ – Example #1 “AAA Batteries”*

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Define Test of Hypothesis:

To find out, we must perform a significance test of

$$H_0: \mu = 30 \text{ hours}$$

$$H_a: \mu > 30 \text{ hours}$$

where μ = the true mean lifetime of the new deluxe AAA batteries.

Check Conditions:

Three conditions should be met before we perform inference for an unknown population mean: **Random, Normal, and Independent.**

The Normal condition for means is

- 1) Population distribution is Normal or
- 2) Sample size is large ($n \geq 30$) or
- 3) We often don't know whether the population distribution is Normal. But if the sample size is large ($n \geq 30$), we can safely carry out a significance test (due to the central limit theorem). If the sample size is small, we should examine the sample data for any obvious departures from Normality, such as skewness and outliers.

■ *The One-Sample t Test for μ* – Example #1 “AAA Batteries”

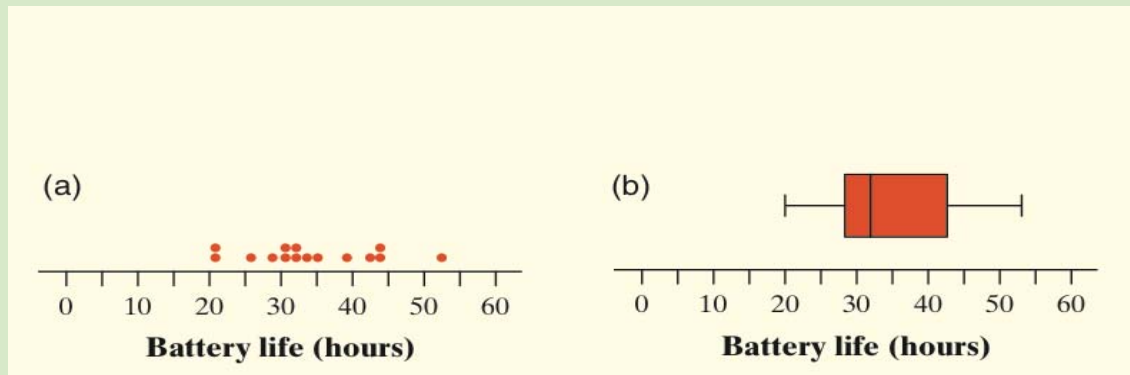
Check Conditions:

Three conditions should be met before we perform inference for an unknown population mean: Random, Normal, and Independent.

- **PLUS** identify σ Known (Z^*) or σ Unknown (T^*)

✓ **Random** The company tests an SRS of 15 new AAA batteries.

✓ **Normal** We don't know if the population distribution of battery lifetimes for the company's new AAA batteries is Normal. With such a small sample size ($n = 15$), we need to inspect the data for any departures from Normality.



The dotplot and boxplot show slight right-skewness but no outliers. ~~The Normal probability plot is close to linear.~~ We should be safe performing a test about the population mean lifetime μ .

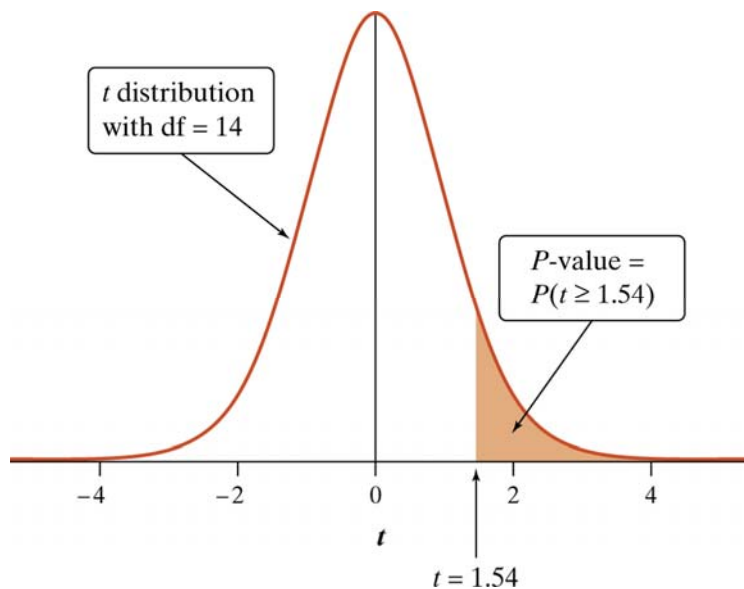
✓ **Independent** Since the batteries are being sampled without replacement, we need to check the *10% condition*: there must be at least $10(15) = 150$ new AAA batteries. This seems reasonable to believe.

■ The One-Sample t Test for μ – Example #1 “AAA Batteries”

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Test: One-Sample t Test for μ with $\alpha=.05$

The sample mean and standard deviation are $\bar{x} = 33.9\text{hrs}$ and $s_x = 9.8\text{hrs}$



$$\text{Test statistic } t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}} = \frac{33.9 - 30}{\frac{9.8}{\sqrt{15}}} = 1.54$$

P-value The P -value is the area to the right of $t=1.54$ under the t distribution curve with $df=15-1=14$

$$P\text{value} = P(t > 1.54) =$$

- Table B: P -value is between 0.05 and 0.10
- Calculator: $\text{tcdf}(1.54, e99, 14) = .0729$

Conclude: Since the P -value=.0729 is large and greater than $\alpha=.05$, we fail to reject H_0 . We don't have enough evidence to conclude the companies new deluxe AAA batteries last more than 30 hours, on average.

■ *The One-Sample t Test for μ – Example #2 “Healthy Streams”*

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EXAMPLE: The level of dissolved oxygen (DO) in a stream or river is an important indicator of the water’s ability to support aquatic life. A researcher measures the DO level at 15 randomly chosen locations along a stream. Here are the results in milligrams per liter:

4.53	5.04	3.29	5.23	4.13	5.50	4.83	4.40
5.42	6.38	4.01	4.66	2.87	5.73	5.55	

A dissolved oxygen level below 5 mg/l puts aquatic life at risk. We want to perform a test at the $\alpha = 0.05$ significance level.

■ *The One-Sample t Test for μ – Example #2 “Healthy Streams”*

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Hypothesis: We want to perform a test at the $\alpha = 0.05$ significance level of

$$H_0: \mu = 5$$

$$H_a: \mu < 5$$

where μ is the actual mean dissolved oxygen level in this stream.

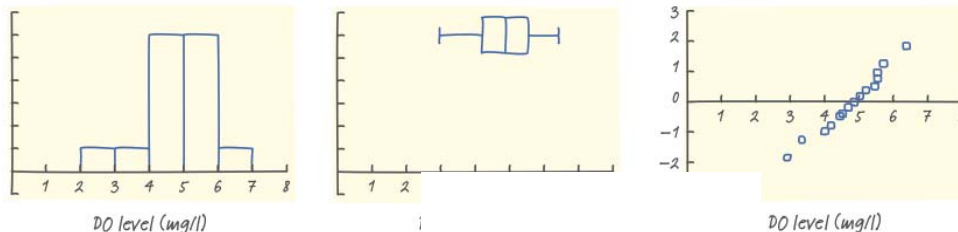
Conditions for selected test : If conditions are met, we should do **a one-sample t test for μ .**

✓ *Unknown σ*

✓ *Random* The researcher measured the DO level at 15 randomly chosen locations.

✓ *Normal* We don't know whether the population distribution of DO levels at all points along the stream is Normal. With such a small sample size ($n = 15$), we need to look at the data to see if it's safe to use t procedures.

The histogram looks roughly symmetric; the boxplot shows no outliers; and the Normal probability plot is fairly linear. With no outliers or strong skewness, the t procedures should be pretty accurate even if the population distribution isn't Normal.

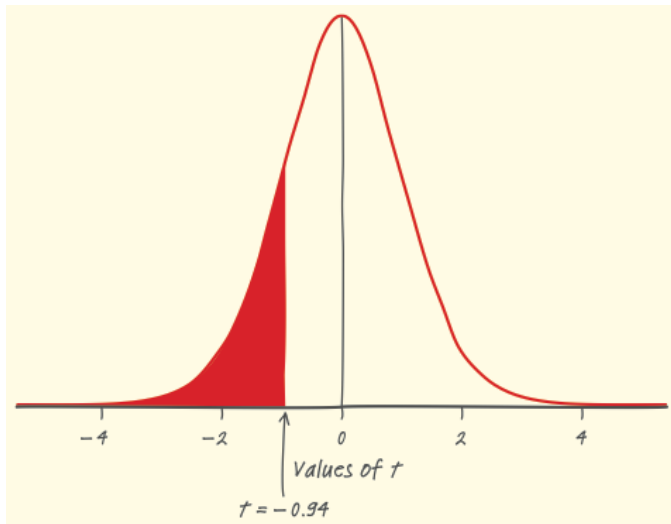


✓ *Independent* There is an infinite number of possible locations along the stream, so it isn't necessary to check the 10% condition. We do need to assume that individual measurements are independent.

■ The One-Sample t Test for μ – Example #2 “Healthy Streams”

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Test: The sample mean and standard deviation are $\bar{x} = 4.771$ and $s_x = 0.9396$



Test statistic
$$t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}} = \frac{4.771 - 5}{\frac{0.9396}{\sqrt{15}}} = -0.94$$

P-value The P -value is the area to the left of $t = -0.94$ under the t distribution curve with $df = 15 - 1 = 14$.

$P\text{value} = P(t < -.94) =$

- Table B - P -value, is between 0.15 and 0.20
- Calculator - $\text{tcdf}(-e99, -.94, 14) = .1816$

Conclude: Since the P -value, is very large we fail to reject H_0 . We don't have enough evidence to conclude that the mean DO level in the stream is less than 5 mg/l.

TYPE 2 ERROR: Since we decided not to reject H_0 , we could have made a Type II error (failing to reject H_0 when H_0 is false). If we did, then the mean dissolved oxygen level μ in the stream is actually less than 5 mg/l, but we didn't detect that with our significance test.

■ 2-Sided t Test for μ – Example #3 “The Hawaii Pineapple Company”

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EXAMPLE: At the Hawaii Pineapple Company, managers are interested in the sizes of the pineapples grown in the company’s fields. Last year, the mean weight of the pineapples harvested from one large field was 31 ounces. A new irrigation system was installed in this field after the growing season. Managers wonder whether this change will affect the mean weight of future pineapples grown in the field. To find out, they select and weigh a random sample of 50 pineapples from this year’s crop. The Minitab

Descriptive Statistics: Weight (oz)

Variable	N	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Weight (oz)	50	31.935	0.339	2.394	26.491	29.990	31.739	34.115	35.547

Since no significance level is given, we’ll use $\alpha = 0.05$.

■ 2-Sided t Test for μ – Example #3 “The Hawaii Pineapple Company”

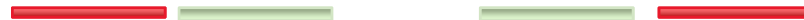
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Determine whether there are any outliers.

Descriptive Statistics: Weight (oz)

Variable	N	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Weight (oz)	50	31.935	0.339	2.394	26.491	29.990	31.739	34.115	35.547



✓ $IQR = Q_3 - Q_1 = 34.115 - 29.990 = 4.125$

✓ Any data value greater than $Q_3 + 1.5(IQR)$ or less than $Q_1 - 1.5(IQR)$ is considered an outlier.

$$Q_3 + 1.5(IQR) = 34.115 + 1.5(4.125) = 40.3025$$

$$Q_1 - 1.5(IQR) = 29.990 - 1.5(4.125) = 23.0825$$

✓ Since the maximum value 35.547 is less than 40.3025 and the minimum value 26.491 is greater than 23.0825, there are no outliers.

■ **2-Sided t Test for μ** – Example #3 “The Hawaii Pineapple Company”

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Test the hypotheses

$$H_0: \mu = 31$$

$$H_a: \mu \neq 31$$

where μ = the mean weight (in ounces) of all pineapples grown in the field this year. Since no significance level is given, we'll use $\alpha = 0.05$.

Check Conditions: If conditions are met, we should do a **one-sample t test for μ** .

✓ *Unknown σ*

✓ *Random* The data came from a random sample of 50 pineapples from this year's crop.

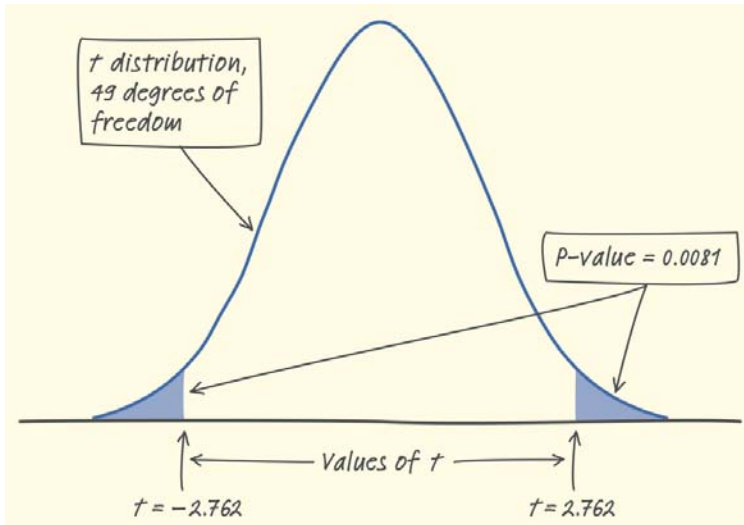
✓ *Normal* We don't know whether the population distribution of pineapple weights this year is Normally distributed. But $n = 50 \geq 30$, so the large sample size (and the fact that there are no outliers) makes it OK to use t procedures.

✓ *Independent* There need to be at least $10(50) = 500$ pineapples in the field because managers are sampling without replacement (*10% condition*). We would expect many more than 500 pineapples in a “large field.”

■ 2-Sided t Test for μ – Example #3 “The Hawaii Pineapple Company”

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TEST: The sample mean and standard deviation are $\bar{x} = 31.935$ and $s_x = 2.394$



$$\text{Test statistic } t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}} = \frac{31.935 - 31}{\frac{2.394}{\sqrt{50}}} = 2.762$$

P-value

Table B: The P -value for this two-sided test is the area under the t distribution curve with $50 - 1 = 49$ degrees of freedom. Since Table B does not have an entry for $df = 49$, we use the more conservative $df = 40$. The upper tail probability is between 0.005 and 0.0025 so the desired P -value is between 0.01 and 0.005.

Upper-tail probability p

df	.005	.0025	.001
30	2.750	3.030	3.385
40	2.704	2.971	3.307
50	2.678	2.937	3.261
	99%	99.5%	99.8%

Confidence level C

Calculator: $P\text{-value} = P(t < -2.76) \text{ or } P(t > 2.76) = .008$
 $\text{tcdf}(2.76, e99, 49) = .004$

Conclude: Since the P -value is very small and it is less than our $\alpha = 0.05$ significance level, we have enough evidence to reject H_0 and conclude that the mean weight of the pineapples in this year's crop is not 31 ounces.

■ Confidence Intervals Give More Information

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Example #3 “The Hawaii Pineapple Company”

Now find a 95% Confidence Interval to estimate the population mean for weight of all the pineapples grown in the field this year.

■ **Example 3:** The Hawaii Pineapple Company

■ **Confidence Intervals Give More Information**



Minitab output for a significance test and confidence interval based on the pineapple data is shown below. The test statistic and P -value match what we got earlier (up to rounding).

One-Sample T: Weight (oz)

Test of $\mu = 31$ vs not = 31

Variable	N	Mean	StDev	SE Mean	95% CI	T	P
Weight (oz)	50	31.935	2.394	0.339	(31.255, 32.616)	2.76	0.008

The 95% confidence interval for the mean weight of all the pineapples grown in the field this year is 31.255 to 32.616 ounces. We are 95% confident that this interval captures the true mean weight μ of this year's pineapple crop.

As with proportions, there is a link between a two-sided test at significance level α and a $100(1 - \alpha)\%$ confidence interval for a population mean μ .

- The pineapples, the 2-sided test at $\alpha = 0.05$ rejects $H_0: \mu = 31$ in favor of $H_a: \mu \neq 31$.
- The corresponding 95% confidence interval does not include 31 as a plausible value of the parameter μ .
- In other words, the test and interval lead to the same conclusion about H_0 .
- **But the confidence interval provides much more information: a set of plausible values for the population mean.**

Confidence Intervals and Two-Sided Tests

The connection between two-sided tests and confidence intervals is even stronger for means than it was for proportions. That's because both inference methods for means use the standard error of the sample mean in the calculations.

$$\text{Test statistic: } t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}} \qquad \text{Confidence interval: } \bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

✓ A two-sided test at significance level α (say, $\alpha = 0.05$) and a $100(1 - \alpha)\%$ confidence interval (a 95% confidence interval if $\alpha = 0.05$) give similar information about the population parameter.

✓ When the two-sided significance test at level α rejects $H_0: \mu = \mu_0$, the $100(1 - \alpha)\%$ confidence interval for μ will not contain the hypothesized value μ_0 .

✓ When the two-sided significance test at level α fails to reject the null hypothesis, the confidence interval for μ will contain μ_0 .



■ Using Tests Wisely

Significance tests are widely used in reporting the results of research in many fields. New drugs require significant evidence of effectiveness and safety. Courts ask about statistical significance in hearing discrimination cases. Marketers want to know whether a new ad campaign significantly outperforms the old one, and medical researchers want to know whether a new therapy performs significantly better. In all these uses, statistical significance is valued because it points to an effect that is unlikely to occur simply by chance.

Carrying out a significance test is often quite simple, especially if you use a calculator or computer. Using tests wisely is not so simple. Here are some points to keep in mind when using or interpreting significance tests.

Statistical Significance and Practical Importance

When a null hypothesis (“no effect” or “no difference”) can be rejected at the usual levels ($\alpha = 0.05$ or $\alpha = 0.01$), there is good evidence of a difference. But that difference may be very small. When large samples are available, even tiny deviations from the null hypothesis will be significant.

■ Using Tests Wisely

Don't Ignore Lack of Significance

There is a tendency to infer that there is no difference whenever a P -value fails to attain the usual 5% standard. In some areas of research, small differences that are detectable only with large sample sizes can be of great practical significance. When planning a study, verify that the test you plan to use has a high probability (power) of detecting a difference of the size you hope to find.

Statistical Inference Is Not Valid for All Sets of Data

Badly designed surveys or experiments often produce invalid results. Formal statistical inference cannot correct basic flaws in the design. Each test is valid only in certain circumstances, with properly produced data being particularly important.

Beware of Multiple Analyses

Statistical significance ought to mean that you have found a difference that you were looking for. The reasoning behind statistical significance works well if you decide what difference you are seeking, design a study to search for it, and use a significance test to weigh the evidence you get. In other settings, significance may have little meaning.

+ Tests About a Population Mean

Summary

In this section, we learned that...

- ✓ Significance tests for the mean μ of a Normal population are based on the sampling distribution of the sample mean. Due to the central limit theorem, the resulting procedures are approximately correct for other population distributions when the sample is large.
- ✓ If we somehow know σ , we can use a z test statistic and the standard Normal distribution to perform calculations.
- ✓ In practice, we typically do not know σ . Then, we use the **one-sample t statistic**

$$t = \frac{\bar{x} - \mu_0}{\frac{s_x}{\sqrt{n}}}$$

with P -values calculated from the t distribution with $n - 1$ degrees of freedom.

+ Tests About a Population Mean

Summary

- ✓ The **one-sample t test** is approximately correct when
 - Random** The data were produced by random sampling or a randomized experiment.
 - Normal** The population distribution is Normal OR the sample size is large ($n \geq 30$).
 - Independent** Individual observations are independent. When sampling without replacement, check that the population is at least 10 times as large as the sample.

PLUS identify σ Known (Z^*) or σ Unknown (T^*)
- ✓ Confidence intervals provide additional information that significance tests do not—namely, a range of plausible values for the parameter μ . A two-sided test of $H_0: \mu = \mu_0$ at significance level α gives the same conclusion as a $100(1 - \alpha)\%$ confidence interval for μ .

+ Tests About a Population Mean

Summary

- ✓ Very small differences can be highly significant (small P -value) when a test is based on a large sample. A statistically significant difference need not be practically important.
- ✓ Lack of significance does not imply that H_0 is true. Even a large difference can fail to be significant when a test is based on a small sample.
- ✓ Significance tests are not always valid. Faulty data collection, outliers in the data, and other practical problems can invalidate a test. Many tests run at once will probably produce some significant results by chance alone, even if all the null hypotheses are true.