

## + Section 9.2

# Tests About a Population Proportion

### Learning Objectives

After this section, you should be able to...

- ✓ CHECK conditions for carrying out a test about a population proportion.
- ✓ CONDUCT a significance test about a population proportion.
- ✓ CONSTRUCT a confidence interval
- ✓ draw a conclusion about for a two-sided test about a population proportion.

## ■ Introduction

Confidence intervals and significance tests are based on the sampling distributions of statistics. That is, both use probability to say what would happen if we applied the inference method many times.

Section 9.1 presented the reasoning of significance tests, including the idea of a  $P$ -value. In this section, we focus on the details of testing a claim about a population proportion.

We'll learn how to perform one-sided and two-sided tests about a population proportion. We'll also see how confidence intervals and two-sided tests are related.

## ■ The One-Sample z Test for a Proportion

The z statistic has approximately the standard Normal distribution when  $H_0$  is true.  $P$ -values therefore come from the standard Normal distribution. Here is a summary of the details for a **one-sample z test for a proportion**.

### One-Sample z Test for a Proportion

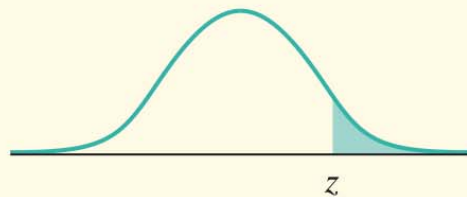
Choose an SRS of size  $n$  from a large population that contains an unknown proportion  $p$  of successes. To test the hypothesis  $H_0 : p = p_0$ , compute the z statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

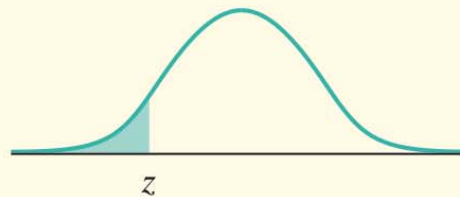
Use this test only when the expected numbers of successes and failures  $np_0$  and  $n(1 - p_0)$  are both at least 10 and the population is at least 10 times as large as the sample.

Find the  $P$ -value by calculating the probability of getting a z statistic this large or larger in the direction specified by the alternative hypothesis  $H_a$ :

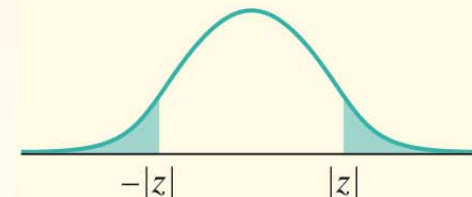
$H_a : p > p_0$



$H_a : p < p_0$



$H_a : p \neq p_0$



- **Example:** The Basketball Player
- ***Carrying Out a Significance Test- A Left Tail Test***

**EXAMPLE:** Suppose a basketball player claimed to be an 80% free-throw shooter. To test this claim, we have him attempt 50 free-throws. He makes 32 of them. His sample proportion of made shots is  $32/50 = 0.64$ . What can we conclude about the claim based on this sample data?

- 1) **State Hypotheses and Sketch Graph:**
- 2) **Check Conditions:**
- 3) **Calculations: Test statistic and P-value:**
- 4) **Conclusion:**

- **Example:** The Basketball Player (continued)
- **Carrying Out a Significance Test- A Left Tail Test**

Recall our basketball player who claimed to be an 80% free-throw shooter. In an SRS of 50 free-throws, he made 32. His sample proportion of made shots,  $32/50 = 0.64$ , is much lower than what he claimed.

Does it provide *convincing evidence* against his claim?

To find out, we must perform a significance test of

$$H_0: p = 0.80$$

$$H_a: p < 0.80$$

where  $p$  = the actual proportion of free throws the shooter makes in the long run.

### Check Conditions:

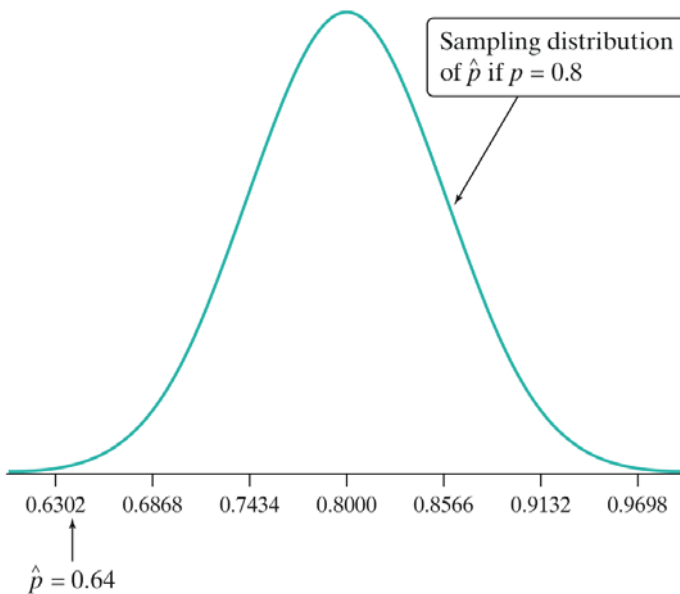
In Chapter 9, we introduced three conditions that should be met before we construct a confidence interval for an unknown population proportion: Random, Normal, and Independent. These same three conditions must be verified before carrying out a significance test.

- ✓ **Random** We can view this set of 50 shots as a simple random sample from the population of all possible shots that the player takes.
- ✓ **Normal** Assuming  $H_0$  is true,  $p = 0.80$ . then  $np = (50)(0.80) = 40$  and  $n(1 - p) = (50)(0.20) = 10$  are both at least 10, so the normal condition is met.
- ✓ **Independent** In our simulation, the outcome of each shot does is determined by a random number generator, so individual observations are independent.

- **Example:** The Basketball Player (continued)
- **Carrying Out a Significance Test- A Left Tail Test**

If the null hypothesis  $H_0 : p = 0.80$  is true, then the player's sample proportion of made free throws in an SRS of 50 shots would vary according to an approximately Normal sampling distribution with mean

$$\mu_{\hat{p}} = p = 0.80 \text{ and standard deviation } \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(0.8)(0.2)}{50}} = 0.0566$$



### Calculations: Test statistic and P-value

A significance test uses sample data to measure the strength of evidence against  $H_0$ . Here are some principles that apply to most tests:

- The test compares a statistic calculated from sample data with the value of the parameter stated by the null hypothesis.
- Values of the statistic far from the null parameter value in the direction specified by the alternative hypothesis give evidence against  $H_0$ .

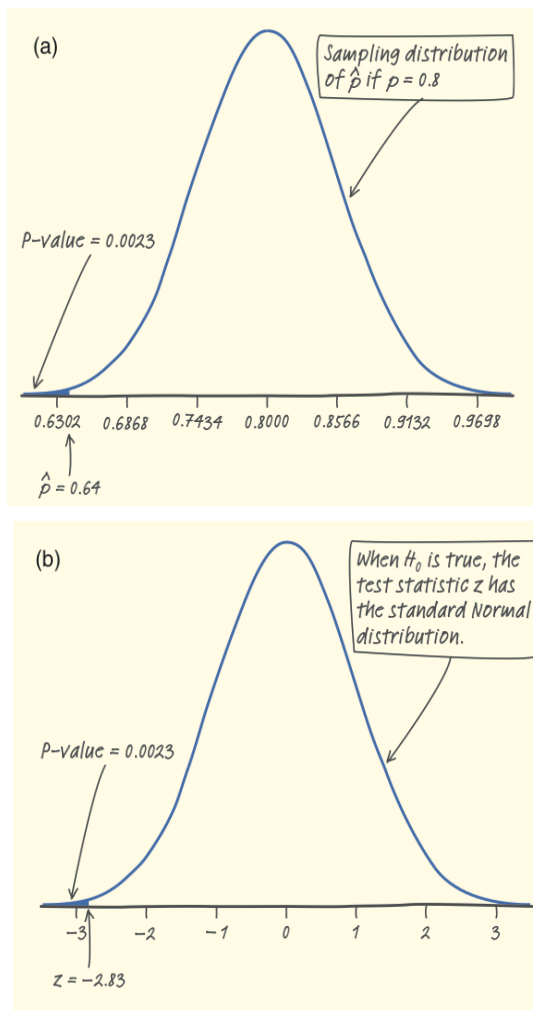
### Definition:

A **test statistic** measures how far a sample statistic diverges from what we would expect if the null hypothesis  $H_0$  were true, in standardized units. That is

$$\text{test statistic} = \frac{\text{statistic} - \text{parameter}}{\text{standard deviation of statistic}}$$

- **Example:** The Basketball Player (continued)
- **Carrying Out a Significance Test- A Left Tail Test**

The test statistic says how far the sample result is from the null parameter value, and in what direction, on a standardized scale. You can use the test statistic to find the  $P$ -value of the test. In our free-throw shooter example, the sample proportion 0.64 is pretty far below the hypothesized value  $H_0: p = 0.80$ . Standardizing, we get



$$\text{test statistic} = \frac{\text{statistic} - \text{parameter}}{\text{standard deviation of statistic}}$$

$$z = \frac{0.64 - 0.80}{0.0566} = -2.83$$

The shaded area under the curve in (a) shows the  $P$ -value. (b) shows the corresponding area on the standard Normal curve, which displays the distribution of the  $z$  test statistic. Using Table A, we find that the  $P$ -value is  $P(z \leq -2.83) = 0.0023$ .

So if  $H_0$  is true, and the player makes 80% of his free throws in the long run, there's only about a 2-in-1000 chance that the player would make as few as 32 of 50 shots.

## ■ SUMMARY: The One-Sample z Test for a Proportion

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### Significance Tests: A Four-Step Process



Tests About a Population Proportion

**State:** What *hypotheses* do you want to test, and at what significance level? Define any *parameters* you use.

**Plan:** Choose the appropriate inference *method*. Check *conditions*.

**Do:** If the conditions are met, perform *calculations*.

- Compute the **test statistic**.
- Find the **P-value**.

**Conclude:** *Interpret* the results of your test in the context of the problem.

When the conditions are met—Random, Normal, and Independent, the sampling distribution of  $\hat{p}$  is approximately Normal with mean

$$\mu_{\hat{p}} = p \text{ and standard deviation } \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}.$$

When performing a significance test, however, the null hypothesis specifies a value for  $p$ , which we will call  $p_0$ . We assume that this value is correct when performing our calculations.



- **Example:** One Potato, Two Potato
- ***Carrying Out a Significance Test - A Right Tail Test***

**EXAMPLE:** A potato-chip producer has just received a truckload of potatoes from its main supplier. If the producer determines that more than 8% of the potatoes in the shipment have blemishes, the truck will be sent away to get another load from the supplier. A supervisor selects a random sample of 500 potatoes from the truck. An inspection reveals that 47 of the potatoes have blemishes. Carry out a significance test at the  $\alpha = 0.10$  significance level. What should the producer conclude?

**Complete the “Test of Significance Template”**

- **Example:** One Potato, Two Potato (continued)
- **Carrying Out a Significance Test - A Right Tail Test**

**State:** We want to perform a test at the  $\alpha = 0.10$  significance level of

$$H_0: p = 0.08$$

$$H_a: p > 0.08$$

where  $p$  is the actual proportion of potatoes in this shipment with blemishes.

**Plan:** If conditions are met, we should do a one-sample  $z$  test for the population proportion  $p$ .

✓ *Random* The supervisor took a random sample of 500 potatoes from the shipment.

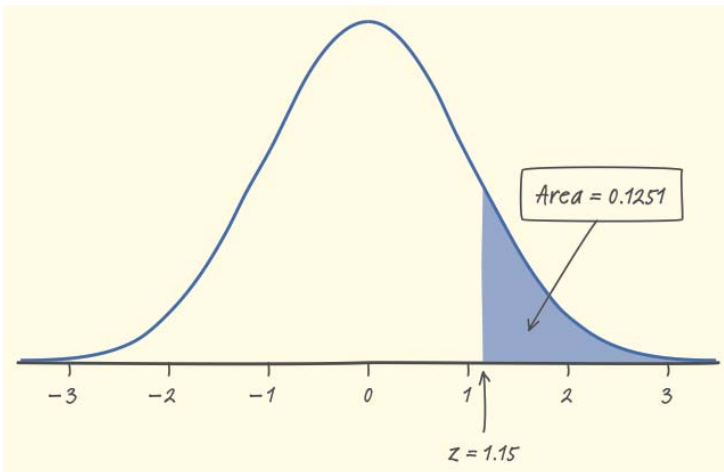
✓ *Normal* Assuming  $H_0: p = 0.08$  is true, the expected numbers of blemished and unblemished potatoes are  $np_0 = 500(0.08) = 40$  and  $n(1 - p_0) = 500(0.92) = 460$ , respectively. Because both of these values are at least 10, we should be safe doing Normal calculations.

✓ *Independent* Because we are sampling without replacement, we need to check the *10% condition*. It seems reasonable to assume that there are at least  $10(500) = 5000$  potatoes in the shipment.

- **Example:** One Potato, Two Potato (continued)
- **Carrying Out a Significance Test - A Right Tail Test**

**Do:** The sample proportion of blemished potatoes is  $\hat{p} = 47/500 = 0.094$ .

**Test statistic** 
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.094 - 0.08}{\sqrt{\frac{0.08(0.92)}{500}}} = 1.15$$



**P-value** Using Table A or `normalcdf(1.15,100)`, the desired  $P$ -value is

$$P(z \geq 1.15) = 1 - 0.8749 = 0.1251$$

**Conclude:** Since our  $P$ -value, 0.1251, is greater than the chosen significance level of  $\alpha = 0.10$ , we fail to reject  $H_0$ . There is not sufficient evidence to conclude that the shipment contains more than 8% blemished potatoes. The producer will use this truckload of potatoes to make potato chips.

- **Example:** High School Student's Smoke?
- ***Carrying Out a Significance Test - A Two Tail Test***

**EXAMPLE:** According to the Centers for Disease Control and Prevention (CDC) Web site, 50% of high school students have never smoked a cigarette. Ty wonders whether this national result holds true in his large, urban high school. For his AP Statistics class project, Ty surveys an SRS of 150 students from his school. He gets responses from all 150 students, and 90 say that they have never smoked a cigarette. What should Ty conclude? Give appropriate evidence to support your answer.

**Complete the “Test of Significance Template”**

- **Example:** High School Student's Smoke?
- **Carrying Out a Significance Test - A Two Tail Test**

**State:** We want to perform a test at the  $\alpha = 0.05$  significance level of

$$H_0: p = 0.50$$

$$H_a: p \neq 0.50$$

where  $p$  is the actual proportion of students in Ty's school who would say they have never smoked cigarettes.

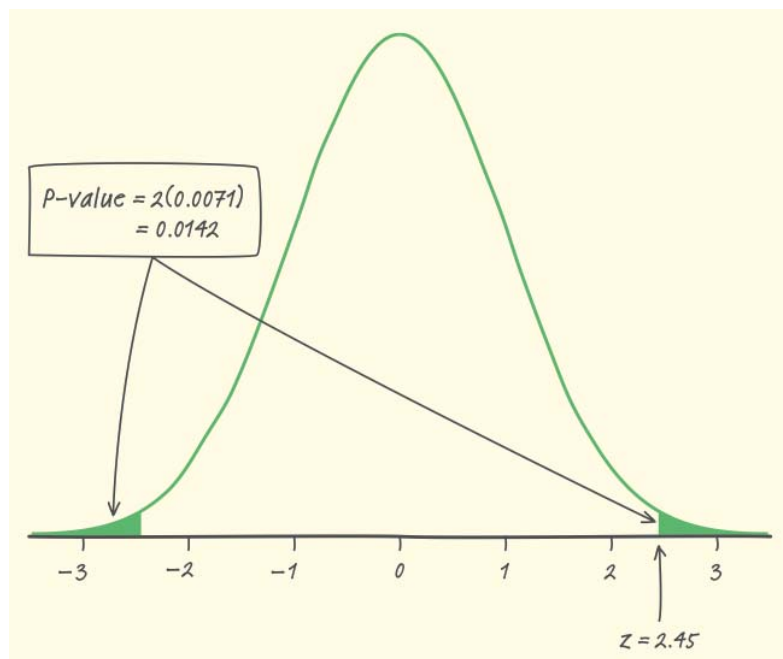
**Plan:** If conditions are met, we should do a one-sample z test for the population proportion  $p$ .

- ✓ *Random* Ty surveyed an SRS of 150 students from his school.
- ✓ *Normal* Assuming  $H_0: p = 0.50$  is true, the expected numbers of smokers and nonsmokers in the sample are  $np_0 = 150(0.50) = 75$  and  $n(1 - p_0) = 150(0.50) = 75$ . Because both of these values are at least 10, we should be safe doing Normal calculations.
- ✓ *Independent* We are sampling without replacement, we need to check the *10% condition*. It seems reasonable to assume that there are at least  $10(150) = 1500$  students at a large high school.

- **Example:** High School Student's Smoke?
- **Carrying Out a Significance Test - A Two Tail Test**

**Do:** The sample proportion is  $\hat{p} = 90/150 = 0.60$

**Test statistic** 
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.60 - 0.50}{\sqrt{\frac{0.50(0.50)}{150}}} = 2.45$$



**P-value** To compute this  $P$ -value, we find the area in one tail and double it. Using Table A or `normalcdf(2.45, 100)` yields  $P(z \geq 2.45) = 0.0071$  (the right-tail area). So the desired  $P$ -value is  $2(0.0071) = 0.0142$ .

**Conclude:** Since our  $P$ -value, 0.0142, is less than the chosen significance level of  $\alpha = 0.05$ , we have sufficient evidence to reject  $H_0$  and conclude that the proportion of students at TyTy's school who say they have never smoked differs from the national result of 0.50.

## ■ Why Confidence Intervals Give More Information

- The result of a significance test is basically a decision to reject  $H_0$  or fail to reject  $H_0$ .
- When we reject  $H_0$ , we're left wondering what the actual proportion  $p$  might be.
- A confidence interval might shed some light on this issue.

1) **Manually Calculate the CI:** Ty found that 90 of an SRS of 150 students said that they had never smoked a cigarette.

Our 95% confidence interval is:

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.60 \pm 1.96 \sqrt{\frac{0.60(0.40)}{150}} = 0.60 \pm 0.078 = (0.522, 0.678)$$

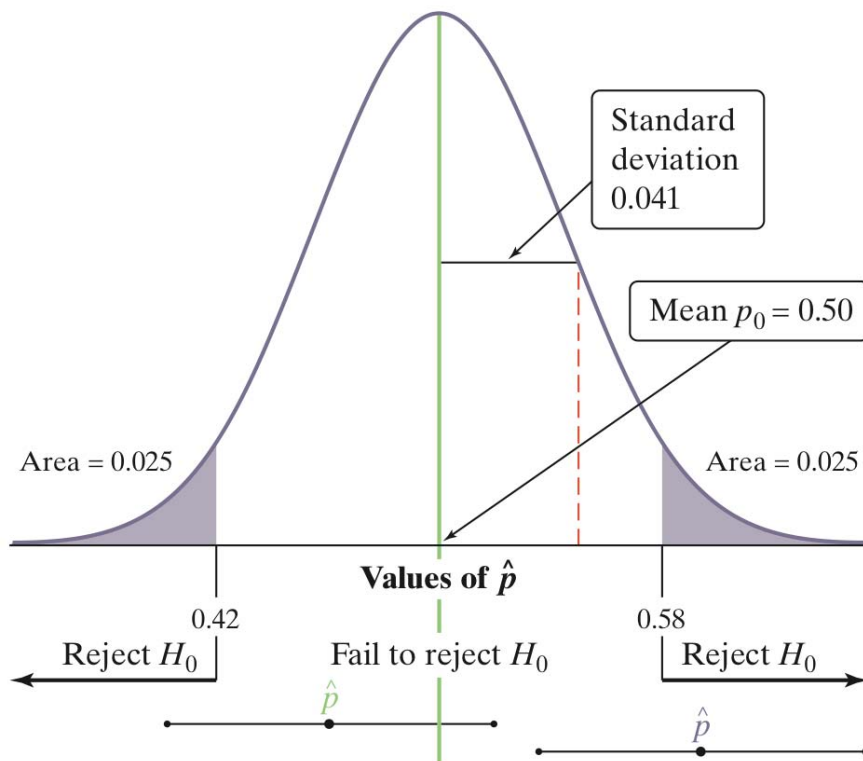
We are 95% confident that the interval from 0.522 to 0.678 captures the true proportion of students at TY's high school who would say that they have never smoked a cigarette.

2) **Calculator Command to find the CI:**

- STAT → TESTS → [A] 1-PropZInterval (x, n, C-Level)
- Where x=# successes; n=sample size; C-Level=confidence level
- **EXAMPLE:** 1-PropZInterval (90, 150, .95) = (.5216, .6784)

## Confidence Intervals and Two-Sided Tests

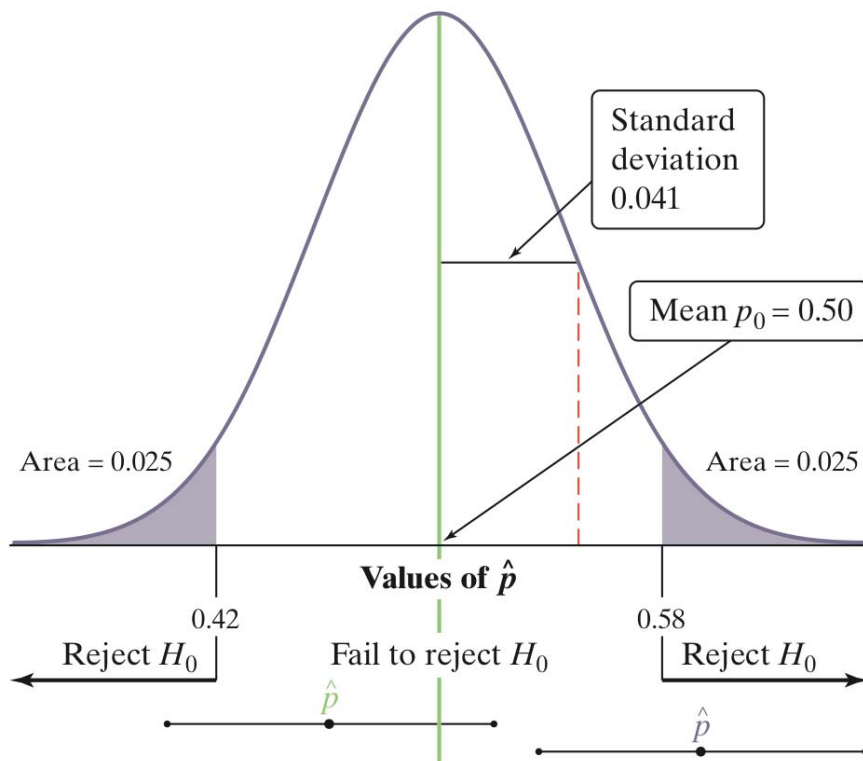
There is a link between confidence intervals and two-sided tests. The 95% confidence interval gives an approximate range of  $p_0$ 's that would not be rejected by a two-sided test at the  $\alpha = 0.05$  significance level. The link isn't perfect because the standard error used for the confidence interval is based on the sample proportion, while the denominator of the test statistic is based on the value  $p_0$  from the null hypothesis.



✓ If the sample proportion falls in the “fail to reject  $H_0$ ” region, like the green value in the figure, the resulting 95% confidence interval would include  $p_0$ . In that case, both the significance test and the confidence interval would be unable to rule out  $p_0$  as a plausible parameter value.



- ✓ A two-sided test at significance level  $\alpha$  (say,  $\alpha = 0.05$ ); and
- ✓ a  $100(1 - \alpha)\%$  confidence interval (a 95% confidence interval if  $\alpha = 0.05$ )
- ✓ give similar information about the population parameter.



- ✓ However, if the sample proportion falls in the “reject  $H_0$ ” region, the resulting 95% confidence interval would not include  $p_0$ . In that case, both the significance test and the confidence interval would provide evidence that  $p_0$  is not the parameter value.

## APPENDIX:

### Reading Notes to supplement the Presentation–

- **READ and STUDY the Following Points and see me with any questions.**

**For addition information see UTUBE Video by Jerry Linch  
One Sample Inference Procedures for Proportions - Confidence Intervals,  
Solving for Sample Size based on Margin of Error - Hypothesis Test  
Construction**

<http://www.youtube.com/watch?v=KSaipum1Ebc&feature=share&list=PLEYFL88U8S9EKRihVcaPfCSc-WVWR6COv>

## + Tests About a Population Proportion

### Summary

In this section, we learned that...

- ✓ As with confidence intervals, you should verify that the three conditions—Random, Normal, and Independent—are met before you carry out a significance test.

- ✓ Significance tests for  $H_0 : p = p_0$  are based on the test statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

with  $P$ -values calculated from the standard Normal distribution.

- ✓ The one-sample  $z$  test for a proportion is approximately correct when
  - (1) the data were produced by random sampling or random assignment;
  - (2) the population is at least 10 times as large as the sample; and
  - (3) the sample is large enough to satisfy  $np_0 \geq 10$  and  $n(1 - p_0) \geq 10$  (that is, the expected numbers of successes and failures are both at least 10).

## + Tests About a Population Proportion

### Summary

In this section, we learned that...

- ✓ Follow the four-step process when you carry out a significance test:
  - STATE:** What hypotheses do you want to test, and at what significance level? Define any parameters you use.
  - PLAN:** Choose the appropriate inference method. Check conditions.
  - DO:** If the conditions are met, perform calculations.
    - Compute the test statistic.
    - Find the P-value.
  - CONCLUDE:** Interpret the results of your test in the context of the problem.
- ✓ Confidence intervals provide additional information that significance tests do not—namely, a range of plausible values for the true population parameter  $p$ . A two-sided test of  $H_0 : p = p_0$  at significance level  $\alpha$  gives roughly the same conclusion as a  $100(1 - \alpha)\%$  confidence interval.