Section 7.5B
Binomial Random Variables

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Learning Objectives

After this section, you should be able to…

- ✓ DETERMINE whether the conditions for a binomial setting are met
- ✓ COMPUTE and INTERPRET probabilities involving binomial random variables
- ✓ CALCULATE the mean and standard deviation of a binomial random variable and INTERPRET these values in context

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Binomial Random Variables

Binomial Settings

When the same chance process is repeated several times, we are often interested in whether a particular outcome does or doesn't happen on each repetition. In some cases, the number of repeated trials is fixed in advance and we are interested in the number of times a particular event (called a "success") occurs. If the trials in these cases are *independent* and each success has an equal chance of occurring, we have a **binomial setting.**

Definition:

A **binomial setting** arises when we perform several independent trials of the same chance process and record the number of times that a particular outcome occurs. The four conditions for a binomial setting are

- **B** • **B**inary? The possible outcomes of each trial can be classified as "success" or "failure."
- **I** • **I**ndependent? Trials must be independent; that is, knowing the result of one trial must not have any effect on the result of any other trial.
- **N** • **N**umber? The number of trials n of the chance process must be fixed in advance.
- **S** • **S**uccess? On each trial, the probability p of success must be the same.

From Blood Types to Aces

Binomial settings and random variables

PROBLEM: Here are three scenarios involving chance behavior. In each case, determine whether the given random variable has a binomial distribution. Justify your answer.

(a) Genetics says that children receive genes from each of their parents independently. Each child of a particular pair of parents has probability 0.25 of having type O blood. Suppose these parents have 5 children. Let X = the number of children with type O blood.

(b) Shuffle a deck of cards. Turn over the first 10 cards, one at a time. Let Y = the number of aces you observe.

(c) Shuffle a deck of cards. Turn over the top card. Put the card back in the deck, and shuffle again. Repeat this process until you get an ace. Let W = the number of cards required.

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From Blood Type to Aces (continued)

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SOLUTION:

(a) To see if this is a binomial setting, we'll check the BINS:

- Binary? "Success" = has type O blood. "Failure" = doesn't have type O blood.
- Independent? Children inherit genes determining blood type independently from each of their parents.
- Number? There are $n = 5$ trials of this chance process.
- Success? The probability of a success is 0.25 on each trial.

This is a binomial setting. Since X counts the number of successes, it is a binomial random variable with parameters $n = 5$ and $p = 0.25$.

(b) Let's check the BINS:

- Binary? "Success" = get an ace. "Failure" = don't get an ace.
- Independent? No. If the first card you turn over is an ace, then the next card is less likely to be an ace because you're not replacing the top card in the deck. Similarly, if the first card isn't an ace, the second card is more likely to be an ace.

Because the trials are not independent, this is not a binomial setting.

From Blood Type to Aces (continued)

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(c) Let's check the BINS:

- Binary? "Success" = get an ace. "Failure" = don't get an ace.
- Independent? Because you are replacing the card in the deck and shuffling each time, the result of one trial does not affect the outcome of any other trial.
- Number? The number of trials is not set in advance. You could get an ace on the first card you turn over, or it may take many cards to get an ace.

Because there is no fixed number of trials, this is not a binomial setting.

Binomial Random Variable

Consider tossing a coin n times. Each toss gives either heads or tails. Knowing the outcome of one toss does not change the probability of an outcome on any other toss. If we define heads as a success, then p is the probability of a head and is 0.5 on any toss.

The number of heads in n tosses is a **binomial random variable *X*.** The probability distribution of X is called a **binomial distribution.**

Definition:

The count X of successes in a binomial setting is a **binomial random variable.** The probability distribution of X is a **binomial distribution** with parameters n and p, where n is the number of trials of the chance process and p is the probability of a success on any one trial. The possible values of X are the whole numbers from 0 to n.

***Note*: When checking the Binomial condition, be sure to check the BINS and make sure you're being asked to count the number of successes in a certain number of trials!**

Binomial Probabilities Example 1

In a binomial setting, we can **define a random variable (say, *X*) as the number of successes** in ***n* independent trials**. **We are interested in finding the probability distribution of *X*.**

Type O Example

Each child of a particular pair of parents has probability 0.25 of having type O blood. Genetics says that children receive genes from each of their parents independently. These parents have 5 children. What is the probability that none of the children has type O blood?

- The count X of children with type O blood is a binomial random variable with $n = 5$ trials and probability $p = 0.25$ of a success on each trial.
- It is reasonable to assume that each child's blood type is independent of each other.
- In this setting, a child with type O blood is a "success" (S) and a child with another blood type is a "failure" (F).
- **What's $P(X = 0)$?**

Example 1

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+ Binomial Example

1) What is the probability that none of the children has type O blood?

P(Success) = (0.25)
P(Failure) = (0.75)

P(X=0) = P(FFFFF) = (0.75)(0.75)(0.75)(0.75)(0.75) = $(0.75)^5$ = .2373

There is about a 24% chance that none of the 5 children have type O blood.

Note: There is only one possible outcome in this scenario to have none of the children have type O blood

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2) What is the probability that ONE of the children has type O blood?

How about $P(X = 1)$? There are several different ways in which exactly 1 of the 5 children could have type O blood. For instance, the first child born might have type O blood, while the remaining 4 children don't have type O blood. The probability that this happens is

$$P(SFFFF) = (0.25)(0.75)(0.75)(0.75)(0.75) = (0.25)(0.75)^4$$

Alternatively, Child 2 could be the one that has type O blood. The corresponding probability is

$$P(FSFFF) = (0.75)(0.25)(0.75)(0.75)(0.75) = (0.25)(0.75)^4$$

There are three more possibilities to consider:

$$P(FFSFF) = (0.75)(0.75)(0.25)(0.75)(0.75) = (0.25)(0.75)^4$$
$$P(FFFSF) = (0.75)(0.75)(0.75)(0.25)(0.75) = (0.25)(0.75)^4$$
$$P(FFFFS) = (0.75)(0.75)(0.75)(0.75)(0.25) = (0.25)(0.75)^4$$

In all, there are five different ways in which exactly 1 child would have type O blood, each with the same probability of occurring. As a result,

2) What is the probability that EXACTLY ONE of the children has type O blood (continued)?

$$P(X = 1) = P(\text{exactly 1 child with type O blood})$$
$$= P(SFFFF) + P(FSFFF) + P(FFSFF) + P(FFFSF) + P(FFFFS)$$
$$= 5(0.25)(0.75)^4 = 0.39551$$

There's about a 40% chance that exactly 1 of the couple's 5 children will hav
type O blood.

3) What is the probability that EXACTLY TWO of the children have type O blood?

Probability of 2 with Type O blood

P(SSFFF) = (0.25)(0.25)(0.75)(0.75)(0.75) = $(0.25)^2(0.75)^3$ = 0.02637

However, there are a number of different arrangements in which 2 out of the 5 children have type O blood:

SSFFF	SFSFF	SFFSF	SFFFS	FSSFF
FSFSF	FSFFS	FFSSF	FFSFS	FFFSS

Verify that in each arrangement, P(X = 2) = $(0.25)^2(0.75)^3$ = 0.02637

Therefore, P(X = 2) = 10$(0.25)^2(0.75)^3$ = 0.2637

There is about a 26% chance that 2 of the 5 children have type O blood.

Binomial Coefficient

Note, in the previous example, *any one arrangement* of 2 S's and 3 F's had the same probability. This is true because no matter what arrangement, we'd multiply together 0.25 twice and 0.75 three times.

We can generalize this for any setting in which we are interested in k successes in n trials. That is,

$$\begin{aligned} P(X=k) &= P(\text{exactly } k \text{ successes in } n \text{ trials}) \\ &= \text{number of arrangements} \cdot p^k (1-p)^{n-k} \end{aligned}$$

Definition:

The number of ways of arranging k successes among n observations is given by the **binomial coefficient**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for k = 0, 1, 2, …, n where

$$n! = n(n-1)(n-2)\bullet\ldots\bullet(3)(2)(1)$$

and 0! = 1.

Binomial Probability

The binomial coefficient counts the number of different ways in which k successes can be arranged among n trials. The binomial probability $P(X = k)$ is this count multiplied by the probability of any one specific arrangement of the k successes.

Binomial Probability – B(n,p)

If X has the binomial distribution with n trials and probability p of success on each trial, the possible values of X are 0, 1, 2, …, n. If k is any one of these values,

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Number of arrangements of k successes

Probability of k successes

Probability of n-k failures

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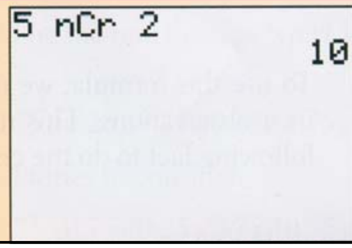
Binomial Probabilities Example 3 (an easier way to find the number of possible outcomes)

TECHNOLOGY CORNER Binomial coefficients on the calculator

To calculate a binomial coefficient like $\binom{5}{2}$ on the TI-83/84 or TI-89, proceed as follows:

TI-83/84: Type 5, press MATH, arrow over to PRB, choose 3:nCr, and press ENTER. Then type 2 and press ENTER again to execute the command 5 nCr 2.



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Now, Find These Binomial Coefficients

$$\left(\frac{5}{0}\right) = \qquad \left(\frac{5}{4}\right) =$$

$$\left(\frac{5}{1}\right) = \qquad \left(\frac{5}{3}\right) = \qquad \left(\frac{5}{5}\right) =$$

Example 4 - Using the Binomial Probability Formula

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- **Type O Blood Example B(5, .25)**
- **Let X = the number of children with type O blood.**

(a) Find the probability that exactly 3 of the children have type O blood.

$$P(X=3)=\binom{5}{3}(0.25)^3(0.75)^2=10(0.25)^3(0.75)^2=0.08789$$

(b) Should the parents be surprised if more than 3 of their children have type O blood?

To answer this, we need to find *P*(*X* > 3).

$$\begin{aligned}P(X>3)&=P(X=4)+P(X=5)\\&=\binom{5}{4}(0.25)^4(0.75)^1+\binom{5}{5}(0.25)^5(0.75)^0\\&=5(0.25)^4(0.75)^1+1(0.25)^5(0.75)^0\\&=0.01465+0.00098=0.01563\end{aligned}$$

Since there is only a 1.5% chance that more than 3 children out of 5 would have Type O blood, the parents should be surprised!

Example 4 –
Using the Calculator to find Binomial Probabilities
Answer Example 4 using the TI84

TECHNOLOGY CORNER Binomial probability on the calculator

There are two handy commands on the TI-83/84 and TI-89 for finding binomial probabilities:

`binompdf(n,p,k)` computes $P(X=k)$
`binomcdf(n,p,k)` computes $P(X \le k)$

These two commands can be found in the distributions menu (2ND/VARS) on the TI-83/84 and in the CATALOG under Flash Apps on the TI-89.

For the parents having $n = 5$ children, each with probability $p = 0.25$ of type O blood:

$P(X = 3) =$ `binompdf(5,0.25,3)` $= 0.087$

Always State B(5, .25)

To find $P(X > 3)$, we used the complement rule:

$P(X > 3) = 1 - P(X \le 3)$
$=$ `1-binomcdf(5,0.25,3)`
$= 0.01563$

```
binompdf(5,.25,3
)
           .087890625
1-binomcdf(5,.25
,3)
              .015625
```

Of course, we could also have done this as

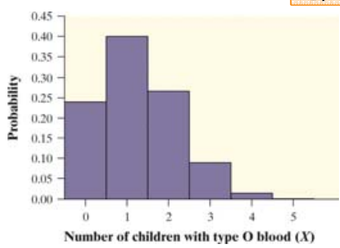
$P(X > 3) = P(X = 4) + P(X = 5)$
$=$ `binompdf(5,0.25,4) + binompdf(5,0.25,5)`
$= 0.01465 + 0.00098 = 0.01563$

EXAMPLE 5: Expected Value and Expected Variance of a Binomial Distribution

- **We describe the probability distribution of a binomial random variable** just like any other distribution – by looking at the **shape**, **center**, and **spread**.
- Describe our probability distribution for:

***X* = number of children with type O blood in a family with 5 children.**

x_i	0	1	2	3	4	5
p_i	0.2373	0.3955	0.2637	0.0879	0.0147	0.00098



Shape: The probability distribution of *X* is skewed to the right. It is more likely to have 0, 1, or 2 children with type O blood than a larger value.

Center: The median number of children with type O blood is 1. Based on our formula for the mean (find the expected value):

$$\mu_X = \sum x_i p_i = (0)(0.2373) + 1(0.39551) + \ldots + (5)(0.00098)$$
$$= 1.25$$

Spread: The variance of *X* is $\sigma_X^2 = \sum (x_i - \mu_X)^2 p_i = (0-1.25)^2(0.2373) + (1-1.25)^2(0.3955) + \ldots +$
$(5-1.25)^2(0.00098) = 0.9375$

The standard deviation of *X* is $\sigma_X = \sqrt{0.9375} = 0.968$

Mean and Standard Deviation of a Binomial Distribution

Notice, the mean μ_X = 1.25 can be found another way. We can use the parameters *n* and *p; and* the method below:

Mean and Standard Deviation of a Binomial Random Variable

If a count *X* has the binomial distribution with number of trials *n* and probability of success *p*, the **mean** and **standard deviation** of *X* are

$$\mu_X = np$$
$$\sigma_X = \sqrt{np(1-p)}$$

***Note:* These formulas work ONLY for binomial distributions. They can't be used for other distributions!**

EXAMPLE 6: Mean and Standard Deviation of a Binomial Distribution

- Type O Example:
- X = number of children with type O blood in a family with 5 children.
- B(5, .25)

Find the mean and standard deviation of X.

Since X is a binomial random variable with parameters n = 5 and p = .25, we can use the formulas for the mean and standard deviation of a binomial random variable.

$$\mu_X = np = 5(.25) = 1.25$$

$$\sigma_X = \sqrt{np(1-p)} = \sqrt{4(.25)(.75)} = .968$$

We'd expect at least 1 of the 5 children to have Type O blood, on average.

If this was repeated many times with groups of 5 children, the number with Type O blood would differ from 1.25 children by an average of 1 child.

Binomial Random Variables

Summary

In this section, we learned that…

- ✓ A **binomial setting** consists of n independent trials of the same chance process, each resulting in a success or a failure, with probability of success p on each trial. The count X of successes is a **binomial random variable**. Its probability distribution is a **binomial distribution**.
- ✓ The **binomial coefficient** counts the number of ways k successes can be arranged among n trials.
- ✓ If X has the binomial distribution with parameters n and p, the possible values of X are the whole numbers 0, 1, 2, . . . , n. The binomial probability of observing k successes in n trials is

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

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Binomial Random Variables

Summary

In this section, we learned that…

- ✓ The **mean** and **standard deviation** of a binomial random variable *X* are

$$\mu_X = np$$

$$\sigma_X = \sqrt{np(1-p)}$$