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Section 7.4
Transforming Random Variables

(DAY 1)

Learning Objectives

After this section, you should be able to…

- ✓ DESCRIBE the effect of performing a linear transformation on a random variable **(DAY 1)**
- ✓ COMBINE random variables and CALCULATE the resulting mean and standard deviation **(DAY 1)**
- ✓ CALCULATE and INTERPRET probabilities involving combinations and Normal random variables **(DAY 2)**
- ✓ **HW: 7-4 Homework Handout -- #'s 24-42**

Linear Transformations

- **In previous Section**, we learned that the mean and standard deviation give us important information about a random variable.
- **In this section**, we'll learn how the mean and standard deviation are affected by transformations on random variables.

ACTIVITY: The Wolf STAT Company:

SUMMARY OF ACTIVITY FINDINGS

1. Adding (or subtracting) a constant, a, to each observation:

- Adds *a* to measures of center and location.
- Does not change the shape or measures of spread.

2. Multiplying (or dividing) each observation by a constant, b:

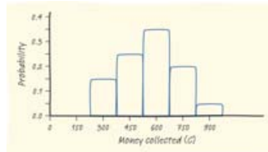
- Multiplies (divides) measures of center and location by *b*.
- Multiplies (divides) measures of spread by |b|.
- Does not change the shape of the distribution.

Linear Transformations – Add/Subtract Constants

Consider Pete's Jeep Tours again. We defined ***C* as the amount of money Pete collects on a randomly selected day.**

Collected c_i	300	450	600	750	900
Probability p_i	0.15	0.25	0.35	0.20	0.05

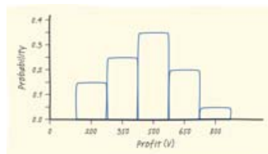
The mean of *C* is \$562.50 and the standard deviation is \$163.50.



It costs Pete \$100 per trip to buy permits, gas, and a ferry pass. The random variable *V* describes the profit Pete makes on a randomly selected day.

Profit v_i	200	350	500	650	800
Probability p_i	0.15	0.25	0.35	0.20	0.05

The mean of *V* is \$462.50 and the standard deviation is \$163.50.



Compare the shape, center, and spread of the two probability distributions.

Linear Transformations - Add/Subtract Constants

How does adding or subtracting a constant affect a random variable?

Effect on a Random Variable of Adding (or Subtracting) a Constant

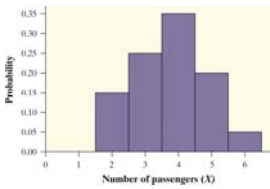
Adding the same number "*a*" (which could be negative) to each value of a random variable:

- "*a*" is added to measures of center and location (mean, median, quartiles, percentiles).
- Does NOT change measures of spread (range, *IQR*, standard deviation).
- Does not change the shape of the distribution.

Linear Transformations – Multiply/Divide Constants

Pete's Jeep Tours offers a popular half-day trip in a tourist area. There must be at least 2 passengers for the trip to run, and the vehicle will hold up to 6 passengers. **Define X as the number of passengers on a randomly selected day**.

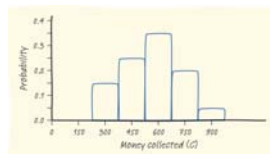
Passengers x_i	2	3	4	5	6
Probability p_i	0.15	0.25	0.35	0.20	0.05



The mean of X is 3.75 and the standard deviation is 1.090.

Pete charges $150 per passenger. **The random variable C describes the amount Pete collects on a randomly selected day.**

Collected c_i	300	450	600	750	900
Probability p_i	0.15	0.25	0.35	0.20	0.05



The mean of C is $562.50 and the standard deviation is $163.50.

Compare the shape, center, and spread of the two probability distributions.

Transforming Random Variables

Linear Transformations – Multiply/Divide Constants

How does multiplying or dividing by a constant affect a random variable?

Effect on a Random Variable of Multiplying (Dividing) by a Constant

Multiplying (or dividing) each value of a random variable by a number b:

- **Multiplies** (divides) **measures of center and location** (mean, median, quartiles, percentiles) by *b.*
- **Multiplies** (divides) **measures of spread** (range, *IQR*, standard deviation) **by $|b|$.**
- **Does not change the shape of the distribution.**

Note: Multiplying a random variable by a constant b multiplies the variance by b^2.

Transforming Random Variables

Linear Transformations - CONCLUSION

Whether we are dealing with data or random variables, the effects of a linear transformation are the same.

Effect on a Linear Transformation on the Mean and Standard Deviation

If $Y = a + bX$ is a linear transformation of the random variable X, then

- The probability distribution of Y has the same shape as the probability distribution of X.
- $\mu_Y = a + b\mu_X$.
- $\sigma_Y = |b|\sigma_X$ (since b could be a negative number).

Transforming Random Variables

Section 7.4
Combining Random Variables

(DAY 2)

Learning Objectives

After this section, you should be able to…

- ✓ CALCULATE and INTERPRET probabilities involving combinations **and** Normal random variables **(DAY 2)**
- ✓ **HW: 7-4 Homework Handout -- #'s 49-63**

Combining Random Variables

So far, we have looked at settings that involve a single random variable. Many interesting statistics problems require us to examine two or more random variables.

Let's investigate the result of adding and subtracting random variables:

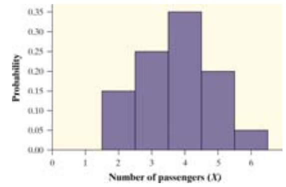
Let X= **the number of passengers on a randomly selected trip with Pete's Jeep Tours.**

Let Y = **the number of passengers on a randomly selected trip with Erin's Adventures.**

Define $T = X + Y$. What are the mean and variance of T?

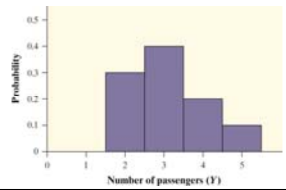
Passengers x_i	2	3	4	5	6
Probability p_i	0.15	0.25	0.35	0.20	0.05

Mean μ_X = 3.75 Standard Deviation σ_X = 1.090



Passengers y_i	**2**	**3**	**4**	**5**
Probability p_i	0.3	0.4	0.2	0.1

Mean μ_Y = 3.10 Standard Deviation σ_Y = 0.943



Combining Random Variables – The Expected Value

How many total passengers can Pete and Erin expect on a randomly selected day?

- **Since Pete expects μ_X = 3.75 and Erin expects μ_Y = 3.10,**
- **They will average a total of 3.75 + 3.10 = 6.85 passengers per trip.**

- **We can generalize this result as follows:**

Mean of the Sum of Random Variables

For any two random variables X and Y, if $T = X + Y$, then the expected value of T is

$$E(T) = \mu_T = \mu_X + \mu_Y$$

In general, the mean of the sum of several random variables is the sum of their means.

Combining Random Variables

Combining Random Variables – Measure of Variability

How much variability is there in the total number of passengers who go on Pete's and Erin's tours on a randomly selected day?

1. **To determine this, we need to find the probability distribution of T.**
2. **The only way to determine the probability for any value of T is if X and Y are INDEPENDENT random variables.**

Definition:

If knowing whether any event involving *X* alone has occurred tells us nothing about the occurrence of any event involving *Y* alone, and vice versa, then *X* and *Y* are **independent random variables.**

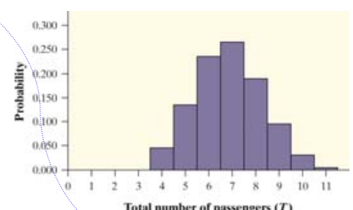
- **Probability models often assume independence when the random variables describe outcomes that appear unrelated to each other.**
- **You should always ask whether the assumption of independence seems reasonable.**
- **In our investigation, it is reasonable to assume *X* and *Y* are independent since the siblings operate their tours in different parts of the country.**

Combining Random Variables

Combining Random Variables – Measure of Variability

Let *T* = *X* + *Y*. Consider all possible combinations of the values of *X*

x_i	p_i	y_i	p_i	$t_i = x_i + y_i$	p_i
2	0.15	2	0.3	4	(0.15)(0.3) = 0.045
2	0.15	3	0.4	5	(0.15)(0.4) = 0.060
2	0.15	4	0.2	6	(0.15)(0.2) = 0.030
2	0.15	5	0.1	7	(0.15)(0.1) = 0.015
3	0.25	2	0.3	5	(0.25)(0.3) = 0.075
3	0.25	3	0.4	6	(0.25)(0.4) = 0.100
3	0.25	4	0.2	7	(0.25)(0.2) = 0.050
3	0.25	5	0.1	8	(0.25)(0.1) = 0.025
4	0.35	2	0.3	6	(0.35)(0.3) = 0.105
4	0.35	3	0.4	7	(0.35)(0.4) = 0.140
4	0.35	4	0.2	8	(0.35)(0.2) = 0.070
4	0.35	5	0.1	9	(0.35)(0.1) = 0.035
5	0.20	2	0.3	7	(0.20)(0.3) = 0.060
5	0.20	3	0.4	8	(0.20)(0.4) = 0.080
5	0.20	4	0.2	9	(0.20)(0.2) = 0.040
5	0.20	5	0.1	10	(0.20)(0.1) = 0.020
6	0.05	2	0.3	8	(0.05)(0.3) = 0.015
6	0.05	3	0.4	9	(0.05)(0.4) = 0.020
6	0.05	4	0.2	10	(0.05)(0.2) = 0.010
6	0.05	5	0.1	11	(0.05)(0.1) = 0.005



Recall: $\mu_T = \mu_X + \mu_Y = 6.85$

$$\sigma_T^2 = \sum (t_i - \mu_T)^2 p_i$$

$= (4 - 6.85)^2(0.045) + \ldots + (11 - 6.85)^2(0.005) =$ **2.0775**

Note: $\sigma_X^2 = 1.1875$ and $\sigma_Y^2 = 0.89$

What do you notice about the variance of *T*?

Combining Random Variables

Combining Random Variables – **Measure of Variability**

As the preceding example illustrates, **when we add two *independent* random variables, their variances add.** *Standard deviations do not add.*

Variance of the Sum of Random Variables

For any two *independent* random variables X and Y, if $T = X + Y$, then the variance of T is

$$\sigma_T^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the sum of several independent random variables is the sum of their variances.

Remember that you can add variances only if the two random variables are independent,

and that you can NEVER add standard deviations!

Note: the more random variables you add, means more variability!!!!!!!!!!!

Combining Random Variables

Combining Random Variables – General Rules

- The same rules apply when we **subtract** Independent Random Variables:

Mean of the Difference of Random Variables

For any two random variables X and Y, if $D = X - Y$, then the expected value of D is

$$E(D) = \mu_D = \mu_X - \mu_Y$$

In general, the mean of the difference of several random variables is the difference of their means. *The order of subtraction is important!*

Variance of the Difference of Random Variables

For any two *independent* random variables X and Y, if $D = X - Y$, then the variance of D is

$$\sigma_D^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the difference of two independent random variables is the sum of their variances.

Combining Random Variables

Combining Normal Random Variables

Combining Random Variables

If a random variable is Normally distributed, we can use its mean and standard deviation to compute probabilities.

- An important fact about Normal random variables is that…

  any sum or difference of independent Normal random variables is also Normally distributed.

Example

Mr. Starnes likes between 8.5 and 9 grams of sugar in his hot tea. Suppose the amount of sugar in a randomly selected packet follows a Normal distribution with mean 2.17 g and standard deviation 0.08 g. If Mr. Starnes selects 4 packets at random, what is the probability his tea will taste right?

Let X = the amount of sugar in a randomly selected packet.
Then, $T = X_1 + X_2 + X_3 + X_4$.
We want to find $P(8.5 \le T \le 9)$.

$$\mu_T = \mu_{X1} + \mu_{X2} + \mu_{X3} + \mu_{X4} = 2.17 + 2.17 + 2.17 + 2.17 = 8.68$$

$$\sigma_T^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \sigma_{X_4}^2 = (0.08)^2 + (0.08)^2 + (0.08)^2 + (0.08)^2 = 0.0256$$

$$\sigma_T = \sqrt{0.0256} = 0.16$$

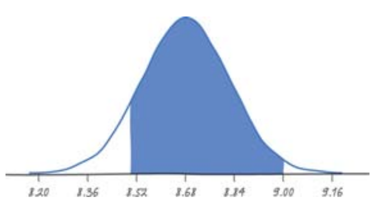
Combining Normal Random Variables

Example (continued)

X = the amount of sugar in a randomly selected packet. $\mu_T = 8.68$
$T = X_1 + X_2 + X_3 + X_4$.

$$\sigma_T = \sqrt{0.0256} = 0.16$$

We want to find $P(8.5 \le T \le 9)$



Method 1: Calculate Z-Scores

$$z = \frac{8.5 - 8.68}{0.16} = -1.13 \quad \text{and} \quad z = \frac{9 - 8.68}{0.16} = 2.00$$

$P(-1.13 \le Z \le 2.00) = 0.9772 – 0.1292 = 0.8480$

Method 2: State N(8.68,.16). Then use ***normalcdf(8.5, 9, 8.68, .16) = 0.8469***

There is about an 85% chance Mr. Starnes's tea will taste right.

Combining Random Variables - TRY THESE

Random variables. Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $2Y + 20$
b) $3X$
c) $0.25X + Y$
d) $X - 5Y$
e) $X_1 + X_2 + X_3$

	Mean	SD
X	80	12
Y	12	3

Kittens. In a litter of seven kittens, three are female. You pick two kittens at random.

a) Create a probability model for the number of male kittens you get.
b) What's the expected number of males?
c) What's the standard deviation?

RANDOM VARIABLE EXAMPLE

X and Y are Independent

$\mu_x = 80$ $\mu_y = 12$

$\sigma_x = 12$ $\sigma_y = 3$

(A) $2Y + 20$:

$\mu = 2(12) + 20$ $\boxed{\mu = 44}$

$SD(2Y + 20) = 2(SD(Y)) = 2(3)$ $\boxed{\sigma = 6}$

↑ Constant does not change variability

(B) $3X$:

$E(3X) = 3 \cdot \mu_x = 3 \cdot 80$ $\boxed{\mu = 240}$

$SD(3X) = 3 \cdot \sigma_x = 3 \cdot 12$ $\boxed{\sigma = 36}$

(C) $.25X + Y$:

$E(.25X + Y) = .25(80) + 12 =$ $\boxed{\mu = 32}$

$SD(.25X + Y) = \sqrt{(.25)^2 \sigma_x^2 + \sigma_y^2}$

$= \sqrt{.0625(144) + 9} = \sqrt{18}$ $\boxed{\sigma = 4.2426}$

(D) $X - 5Y$:

$E(X - 5Y) = \mu_x - 5\mu_y = 80 - 5(12) =$ $\boxed{\mu = 20}$

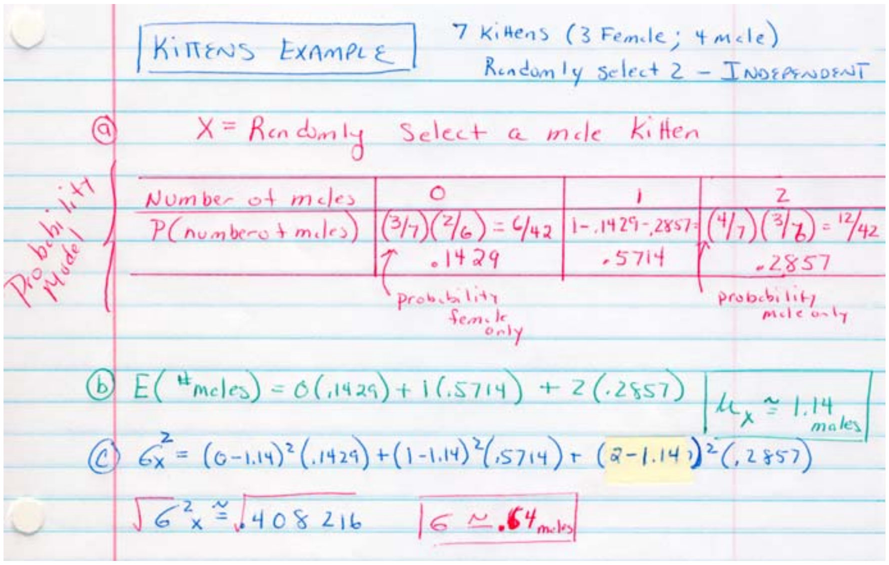
$SD(X - 5Y) = \sqrt{(\sigma_x)^2 + (-5)^2(\sigma_y)^2} = \sqrt{12^2 + 25(9)} = \sqrt{369}$

$\boxed{\sigma = 19.209}$

(E) $X_1 + X_2 + X_3$:

$E(X_1 + X_2 + X_3) = 80 + 80 + 80$ $\boxed{\mu = 240}$

$SD(X_1 + X_2 + X_3) = \sqrt{12^2 + 12^2 + 12^2} = \sqrt{432}$ $\boxed{\sigma = 20.78}$



Transforming and Combining Random Variables

Summary

In this section, we learned that…

- ✓ Adding a constant *a* (which could be negative) to a random variable increases (or decreases) the mean of the random variable by *a* but does not affect its standard deviation or the shape of its probability distribution.
- ✓ Multiplying a random variable by a constant *b* (which could be negative) multiplies the mean of the random variable by *b* and the standard deviation by |*b*| but does not change the shape of its probability distribution.
- ✓ A **linear transformation** of a random variable involves adding a constant *a*, multiplying by a constant *b*, or both. If we write the linear transformation of *X* in the form $Y = a + bX$, the following about are true about *Y*:
  - ✓ **Shape:** same as the probability distribution of *X*.
  - ✓ **Center:** $\mu_Y = a + b\mu_X$
  - ✓ **Spread:** $\sigma_Y = |b|\sigma_X$

Transforming and Combining Random Variables

Summary

In this section, we learned that…

- If X and Y are any two random variables,

$$\mu_{X \pm Y} = \mu_X \pm \mu_Y$$

- If X and Y are **independent random variables**

$$\sigma^2_{X \pm Y} = \sigma^2_X + \sigma^2_Y$$

- The sum or difference of independent Normal random variables follows a Normal distribution.