Chapter 5: Probability: What are the Chances?

Section 5.1
Randomness, Probability, and Simulation

The Practice of Statistics, 4th edition – For AP*
STARNES, YATES, MOORE

Chapter 5
Probability: What Are the Chances?

- **5.1 Randomness, Probability, and Simulation**
- 5.2 Probability Rules
- 5.3 Conditional Probability and Independence

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Section 5.1
Randomness, Probability, and Simulation

Learning Objectives

After this section, you should be able to…

- ✓ DESCRIBE the idea of probability
- ✓ DESCRIBE myths about randomness
- ✓ DESIGN and PERFORM simulations

The Idea of Probability

Randomness, Probability, and Simulation

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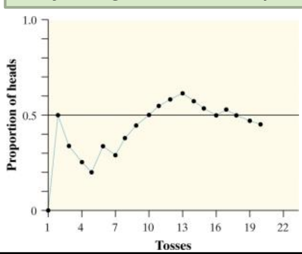
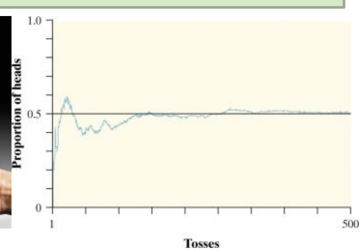
Chance behavior is unpredictable in the short run, but has a regular and predictable pattern in the long run.

The **law of large numbers** says that if we observe more and more repetitions of any chance process, the proportion of times that a specific outcome occurs approaches a single value.

Definition:

The **probability** of any outcome of a chance process is a number between 0 (never occurs) and 1(always occurs) that describes the proportion of times the outcome would occur in a very long series of repetitions.





Myths about Randomness

"So the law of averages doesn't guarantee me a girl after seven straight boys, but can't I at least get a group discount on the delivery fee?"

Simulation

The imitation of chance behavior, based on a model that accurately reflects the situation, is called a **simulation**.

Performing a Simulation

State: What is the question of interest about some chance process?

Plan: Describe how to use a chance device to imitate one repetition of the process. Explain clearly how to identify the outcomes of the chance process and what variable to measure.

Do: Perform many repetitions of the simulation.

Conclude: Use the results of your simulation to answer the question of interest.

We can use physical devices, random numbers (e.g. Table D), and technology to perform simulations.

Example: Golden Ticket Parking Lottery

Read the example on page 290.

What is the probability that a fair lottery would result in two winners from the AP Statistics class?

Students	Labels
AP Statistics Class	01-28
Other	29-95
Skip numbers from 96-00	

Reading across row 139 in Table D, look at pairs of digits until you see two different labels from 01-95. Record whether or not both winners are members of the AP Statistics Class.

55 | 58	**89 | 94**	**04 | 70**	**70 | 84**	**10|98|43**	**56 | 35**	**69 | 34**	**48 | 39**	**45 | 17**
X | X	X | X	✓ | X	X | X	✓|Sk|X	X | X	X | X	X | X	X | ✓
No	No	No	No	No	No	No	No	No

19 | 12	**97|51|32**	**58 | 13**	**04 | 84**	**51 | 44**	**72 | 32**	**18 | 19**	**40|00|36**	**00|24|28**
✓ | ✓	Sk|X|X	X | ✓	✓ | X	X | X	X | X	✓ | ✓	X|Sk|X	Sk|✓|✓
Yes	No	No	No	No	No	**Yes**	No	**Yes**

Based on 18 repetitions of our simulation, both winners came from the AP Statistics class 3 times, so the probability is estimated as 16.67%.

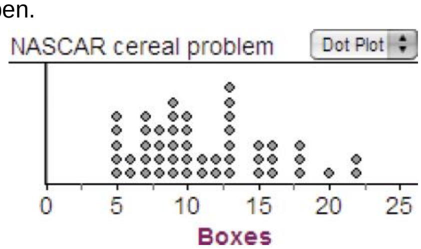
Example: NASCAR Cards and Cereal Boxes

Read the example on page 291.

What is the probability that it will take 23 or more boxes to get a full set of 5 NASCAR collectible cards?

Driver	Label
Jeff Gordon	1
Dale Earnhardt, Jr.	2
Tony Stewart	3
Danica Patrick	4
Jimmie Johnson	5

Use randInt(1,5) to simulate buying one box of cereal and looking at which card is inside. Keep pressing Enter until we get all five of the labels from 1 to 5. Record the number of boxes we had to open.



3 5 2 1 5 2 3 5 4 **9 boxes**

4 3 5 3 5 1 1 1 5 3 1 5 4 5 2 **15 boxes**

5 5 5 2 4 1 2 1 5 3 **10 boxes**

We never had to buy more than 22 boxes to get the full set of cards in 50 repetitions of our simulation. Our estimate of the probability that it takes 23 or more boxes to get a full set is roughly 0.

Section 5.1
Randomness, Probability, and Simulation

Summary

In this section, we learned that…

- ✓ A chance process has outcomes that we cannot predict but have a regular distribution in many distributions.
- ✓ The **law of large numbers** says the proportion of times that a particular outcome occurs in many repetitions will approach a single number.
- ✓ The long-term relative frequency of a chance outcome is its **probability** between 0 (never occurs) and 1 (always occurs).
- ✓ Short-run regularity and the law of averages are myths of probability.
- ✓ A **simulation** is an imitation of chance behavior.

Section 5.2
Probability Rules

Learning Objectives

After this section, you should be able to…

- ✓ DESCRIBE chance behavior with a probability model
- ✓ DEFINE and APPLY basic rules of probability
- ✓ DETERMINE probabilities from two-way tables
- ✓ CONSTRUCT Venn diagrams and DETERMINE probabilities

Probability Models

Probability Rules

In Section 5.1, we used simulation to imitate chance behavior. Fortunately, we don't have to always rely on simulations to determine the probability of a particular outcome.

Descriptions of chance behavior contain two parts:

Definition:

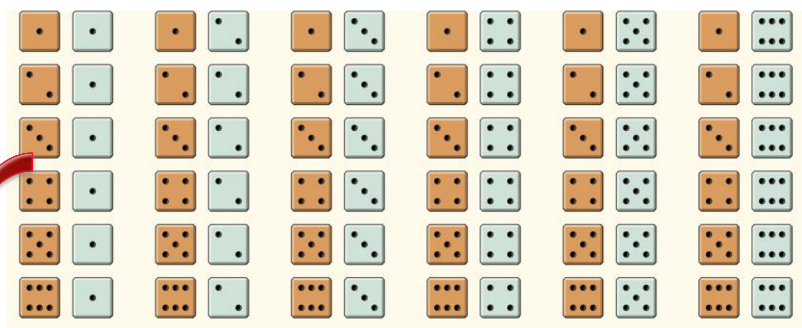
The **sample space *S*** of a chance process is the set of all possible outcomes.

A **probability model** is a description of some chance process that consists of two parts: a sample space *S* and a probability for each outcome.

Example: Roll the Dice

Probability Rules

Give a probability model for the chance process of rolling two fair, six-sided dice – one that's red and one that's green.



Sample Space 36 Outcomes

Since the dice are fair, each outcome is equally likely. Each outcome has probability 1/36.

Probability Models

Probability models allow us to find the probability of any collection of outcomes.

Definition:

An **event** is any collection of outcomes from some chance process. That is, an event is a subset of the sample space. Events are usually designated by capital letters, like *A, B, C,* and so on.

If *A* is any event, we write its probability as P(*A*).

In the dice-rolling example, suppose we define event *A* as "sum is 5."

There are 4 outcomes that result in a sum of 5.
Since each outcome has probability 1/36, P(*A*) = 4/36.

Suppose event *B* is defined as "sum is not 5." What is P(*B*)? P(*B*) = 1 – 4/36
= 32/36

Basic Rules of Probability

All probability models must obey the following rules:

- **The probability of any event is a number between 0 and 1.**
- **All possible outcomes together must have probabilities whose sum is 1.**
- **If all outcomes in the sample space are equally likely, the probability that event *A* occurs can be found using the formula**

$$P(A) = \frac{\text{number of outcomes corresponding to event } A}{\text{total number of outcomes in sample space}}$$

- **The probability that an event does not occur is 1 minus the probability that the event does occur.**
- **If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities.**

Definition:

Two events are **mutually exclusive (disjoint)** if they have no outcomes in common and so can never occur together.

Basic Rules of Probability

- For any event A, $0 \le P(A) \le 1$.
- If S is the sample space in a probability model,

  $P(S) = 1$.
- In the case of equally likely outcomes,

$$P(A) = \frac{\text{number of outcomes corresponding to event } A}{\text{total number of outcomes in sample space}}$$

- **Complement rule:** $P(A^C) = 1 - P(A)$
- **Addition rule for mutually exclusive events:** If A and B are mutually exclusive,

  $P(A \text{ or } B) = P(A) + P(B)$.

Probability Rules

Example: Distance Learning

Distance-learning courses are rapidly gaining popularity among college students. Randomly select an undergraduate student who is taking distance-learning courses for credit and record the student's age. Here is the probability model:

Age group (yr):	18 to 23	24 to 29	30 to 39	40 or over
Probability:	0.57	0.17	0.14	0.12

(a) Show that this is a legitimate probability model.

Each probability is between 0 and 1 and $0.57 + 0.17 + 0.14 + 0.12 = 1$

(b) Find the probability that the chosen student is not in the traditional college age group (18 to 23 years).

P(not 18 to 23 years) = 1 – P(18 to 23 years)
= 1 – 0.57 = 0.43

Probability Rules

Two-Way Tables and Probability

Probability Rules

When finding probabilities involving two events, a two-way table can display the sample space in a way that makes probability calculations easier.

Consider the example on page 303. Suppose we choose a student at random. Find the probability that the student

	Pierced Ears?		
Gender	**Yes**	**No**	**Total**
Male	19	71	**90**
Female	84	4	**88**
Total	**103**	**75**	**178**

(a) has pierced ears.

(b) is a male with pierced ears.

(c) is a male or has pierced ears.

Define events *A*: is male and *B*: has pierced ears.

(a) Each student is equally likely to be chosen. 103 students have pierced ears. So, *P(*pierced ears) = *P*(*B*) = 103/178.

(b) We want to find *P*(male *and* pierced ears), that is, *P*(*A* and *B*). Look at the intersection of the "Male" row and "Yes" column. There are 19 males with pierced ears. So, *P*(*A* and *B*) = 19/178.

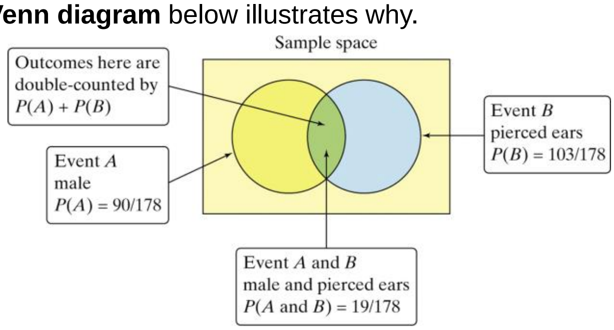
(c) We want to find *P*(male *or* pierced ears), that is, *P*(*A* or *B*). There are 90 males in the class and 103 individuals with pierced ears. However, 19 males have pierced ears – don't count them twice! *P*(*A* or *B*) = (19 + 71 + 84)/178. So, *P*(*A* or B*)* = 174/178

Two-Way Tables and Probability

Probability Rules

Note, the previous example illustrates the fact that we can't use the addition rule for mutually exclusive events unless the events have no outcomes in common.

The **Venn diagram** below illustrates why.



General Addition Rule for Two Events

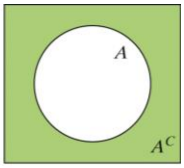
If *A* and *B* are any two events resulting from some chance process, then

P(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*)

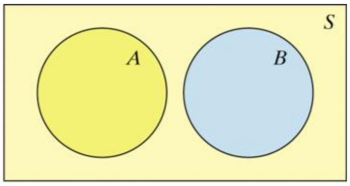
Venn Diagrams and Probability

Because Venn diagrams have uses in other branches of mathematics, some standard vocabulary and notation have been developed.

The complement A^C contains exactly the outcomes that are not in *A*.



The events *A* and *B* are mutually exclusive (disjoint) because they do not overlap. That is, they have no outcomes in common.

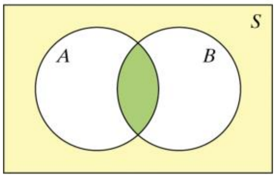


Probability Rules

Venn Diagrams and Probability

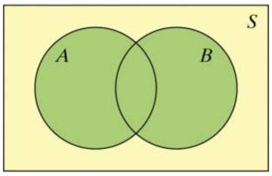
The intersection of events *A* and *B* ($A \cap B$) is the set of all outcomes in both events *A and B*.

$A \cap B$



The union of events *A* and *B* ($A \cup B$) is the set of all outcomes in either event *A or B.*

$A \cup B$



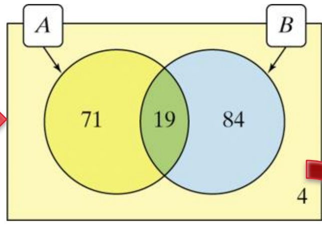
Hint: To keep the symbols straight, remember ∪ for union and ∩ for intersection.

Probability Rules

Venn Diagrams and Probability

Recall the example on gender and pierced ears. We can use a Venn diagram to display the information and determine probabilities.

Gender	Pierced Ears? Yes	No	Total
Male	19	71	**90**
Female	84	4	**88**
Total	**103**	**75**	**178**



Define events *A*: is male and *B*: has pierced ears.

Region in Venn diagram	In words	In symbols	Count
In the intersection of two circles	Male and pierced ears	$A \cap B$	19
Inside circle *A*, outside circle *B*	Male and no pierced ears	$A \cap B^C$	71
Inside circle *B*, outside circle *A*	Female and pierced ears	$A^C \cap B$	84
Outside both circles	Female and no pierced ears	$A^C \cap B^C$	4

Section 5.2 Probability Rules

Summary

In this section, we learned that…

- ✓ A **probability model** describes chance behavior by listing the possible outcomes in the **sample space *S*** and giving the probability that each outcome occurs.
- ✓ An **event** is a subset of the possible outcomes in a chance process.
- ✓ For any event A, $0 \le P(A) \le 1$
- ✓ $P(S) = 1$, where S = the sample space
- ✓ If all outcomes in S are equally likely, $P(A) = \frac{\text{number of outcomes corresponding to event } A}{\text{total number of outcomes in sample space}}$
- ✓ $P(A^C) = 1 - P(A)$, where A^C is the **complement** of event A; that is, the event that A does not happen.

Section 5.2
Probability Rules

Summary

In this section, we learned that…

- ✓ Events *A* and *B* are **mutually exclusive (disjoint)** if they have no outcomes in common. If *A* and *B* are disjoint, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*).
- ✓ A **two-way table** or a **Venn diagram** can be used to display the sample space for a chance process.
- ✓ The **intersection ($A \cap B$)** of events *A* and *B* consists of outcomes in both *A* and *B*.
- ✓ The **union ($A \cup B$)** of events *A* and *B* consists of all outcomes in event *A*, event *B*, or both.
- ✓ The **general addition rule** can be used to find *P*(*A* or *B*):

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Section 5.3
Conditional Probability and Independence

Learning Objectives

After this section, you should be able to…

- ✓ DEFINE conditional probability
- ✓ COMPUTE conditional probabilities
- ✓ DESCRIBE chance behavior with a tree diagram
- ✓ DEFINE independent events
- ✓ DETERMINE whether two events are independent
- ✓ APPLY the general multiplication rule to solve probability questions

What is Conditional Probability?

The probability we assign to an event can change if we know that some other event has occurred. This idea is the key to many applications of probability.

When we are trying to find the probability that one event will happen under the *condition* that some other event is already known to have occurred, we are trying to determine a **conditional probability.**

Definition:

The probability that one event happens given that another event is already known to have happened is called a **conditional probability.** Suppose we know that event *A* has happened. Then the probability that event *B* happens *given* that event *A* has happened is denoted by *P(B | A).*

Read | as "given that" or "under the condition that"

Conditional Probability and Independence

Example: Grade Distributions

Consider the two-way table on page 314. Define events

E: the grade comes from an EPS course, and

L: the grade is lower than a B.

School	A	B	Below B	Total
	Grade Level			
Liberal Arts	2,142	1,890	2,268	6300
Engineering and Physical Sciences	368	432	800	1600
Health and Human Services	882	630	588	2100
Total	3392	2952	3656	*10000*

Find *P*(*L*)

***P*(*L*) = 3656 / 10000 = 0.3656**

Find *P*(*E* | *L*)

***P*(*E* | *L*) = 800 / 3656 = 0.2188**

Find *P*(*L* | *E*)

***P*(*L*| *E*) = 800 / 1600 = 0.5000**

Conditional Probability and Independence

Conditional Probability and Independence

When knowledge that one event has happened does not change the likelihood that another event will happen, we say the two events are **independent.**

Definition:

Two events *A* and *B* are **independent** if the occurrence of one event has no effect on the chance that the other event will happen. In other words, events *A* and *B* are independent if

$P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

Example:

	Dominant Hand		
Gender	Right	Left	Total
Male	20	3	**23**
Female	23	4	**27**
Total	**43**	**7**	**50**

Are the events "male" and "left-handed" independent? Justify your answer.

***P*(left-handed | male) = 3/23 = 0.13**

***P*(left-handed) = 7/50 = 0.14**

These probabilities are not equal, therefore the events "male" and "left-handed" are not independent.

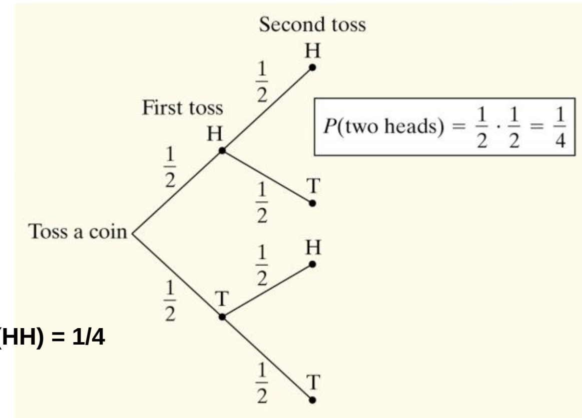
Tree Diagrams

Conditional Probability and Independence

We learned how to describe the sample space *S* of a chance process in Section 5.2. Another way to model chance behavior that involves a sequence of outcomes is to construct a **tree diagram**.

Consider flipping a coin twice.

What is the probability of getting two heads?



Sample Space:
HH HT TH TT

So, *P*(two heads) = *P*(HH) = 1/4

General Multiplication Rule

Conditional Probability and Independence

The idea of multiplying along the branches in a tree diagram leads to a general method for finding the probability $P(A \cap B)$ that two events happen together.

General Multiplication Rule

The probability that events *A* and *B* both occur can be found using the **general multiplication rule**

$$P(A \cap B) = P(A) \bullet P(B \mid A)$$

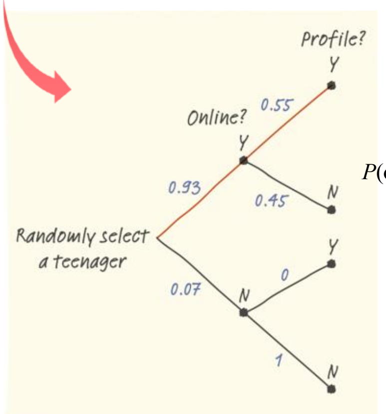
where $P(B \mid A)$ is the conditional probability that event *B* occurs given that event *A* has already occurred.

Example: Teens with Online Profiles

Conditional Probability and Independence

The Pew Internet and American Life Project finds that 93% of teenagers (ages 12 to 17) use the Internet, and that 55% of online teens have posted a profile on a social-networking site.

What percent of teens are online *and* have posted a profile?



$$P(\text{online}) = 0.93$$

$$P(\text{profile} \mid \text{online}) = 0.55$$

$$P(\text{online and have profile}) = P(\text{online}) \cdot P(\text{profile} \mid \text{online})$$

$$= (0.93)(0.55)$$

$$= 0.5115$$

51.15% of teens are online *and* have posted a profile.

Example: Who Visits YouTube?

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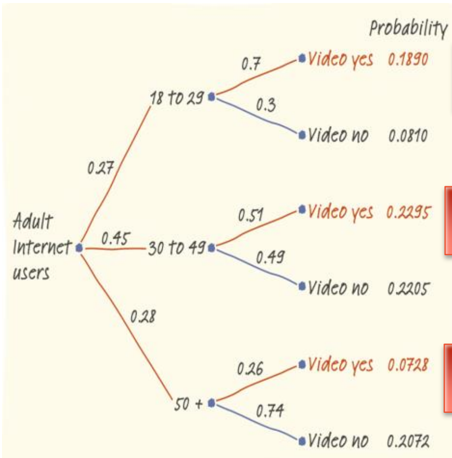
See the example on page 320 regarding adult Internet users.

What percent of all adult Internet users visit video-sharing sites?



***P*(video yes ∩ 18 to 29) = 0.27 • 0.7 =0.1890**

***P*(video yes ∩ 30 to 49) = 0.45 • 0.51 =0.2295**

***P*(video yes ∩ 50 +) = 0.28 • 0.26 =0.0728**

***P*(video yes) = 0.1890 + 0.2295 + 0.0728 = 0.4913**

Independence: A Special Multiplication Rule

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When events *A* and *B* are independent, we can simplify the general multiplication rule since $P(B|\ A) = P(B)$.

Definition:

Multiplication rule for independent events

If *A* and *B* are independent events, then the probability that *A* and *B* both occur is

$$P(A \cap B) = P(A) \cdot P(B)$$

Example:

Following the Space Shuttle *Challenger* disaster, it was determined that the failure of O-ring joints in the shuttle's booster rockets was to blame. Under cold conditions, it was estimated that the probability that an individual O-ring joint would function properly was 0.977. Assuming O-ring joints succeed or fail independently, what is the probability all six would function properly?

***P*(joint1 OK and joint 2 OK and joint 3 OK and joint 4 OK and joint 5 OK and joint 6 OK)**
=P(joint 1 OK) • P(joint 2 OK) • … • P(joint 6 OK)
=(0.977)(0.977)(0.977)(0.977)(0.977)(0.977) = 0.87

Conditional Probability and Independence

Conditional Probability and Independence

Calculating Conditional Probabilities

If we rearrange the terms in the general multiplication rule, we can get a formula for the conditional probability $P(B \mid A)$.

General Multiplication Rule

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

Conditional Probability Formula

To find the conditional probability P(B | A), use the formula

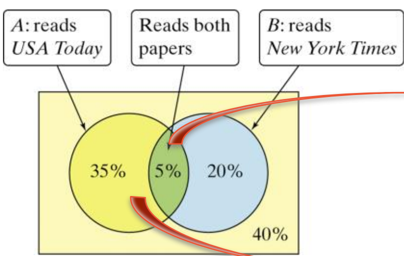
= ________

Conditional Probability and Independence

Example: Who Reads the Newspaper?

In Section 5.2, we noted that residents of a large apartment complex can be classified based on the events *A*: reads *USA Today* and *B*: reads the *New York Times*. The Venn Diagram below describes the residents.

What is the probability that a randomly selected resident who reads *USA Today* also reads the *New York Times*?



$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = 0.05$$

$$P(A) = 0.40$$

$$P(B \mid A) = \frac{0.05}{0.40} = 0.125$$

There is a 12.5% chance that a randomly selected resident who reads *USA Today* also reads the *New York Times*.

Section 5.3
Conditional Probability and Independence

Summary

In this section, we learned that…

- ✓ If one event has happened, the chance that another event will happen is a **conditional probability**. $P(B|A)$ represents the probability that event B occurs given that event A has occurred.
- ✓ Events A and B are **independent** if the chance that event B occurs is not affected by whether event A occurs. If two events are mutually exclusive (disjoint), they cannot be independent.
- ✓ When chance behavior involves a sequence of outcomes, a **tree diagram** can be used to describe the sample space.
- ✓ The **general multiplication rule** states that the probability of events A and B occurring together is $P(A \cap B)=P(A) \cdot P(B|A)$
- ✓ In the special case of *independent* events, $P(A \cap B)=P(A) \cdot P(B)$
- ✓ The **conditional probability formula** states $P(B|A) = P(A \cap B) / P(A)$